Uniform Beam Distributions of Charged Particle Beams

Workshop on Non-linear beam expander systems in high-power accelerator facilities
ISA, Department of Physics and Astronomy, Aarhus University, Aarhus, Denmark
26th and 27th March 2012

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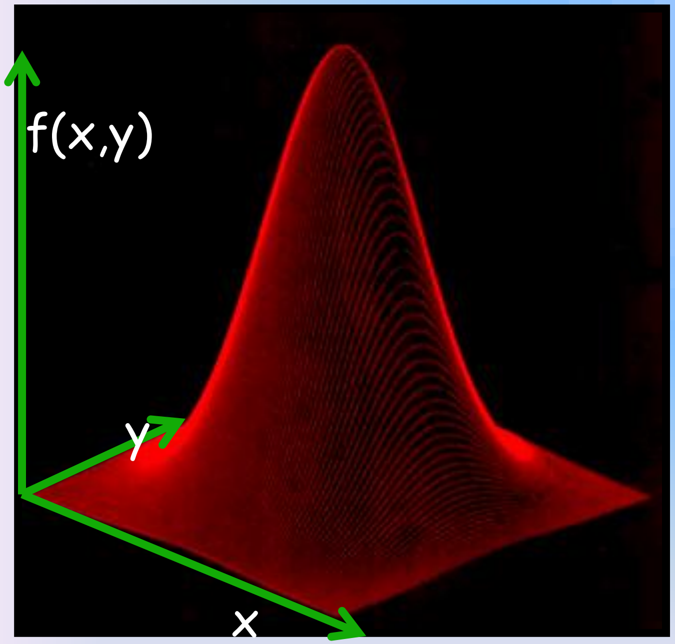

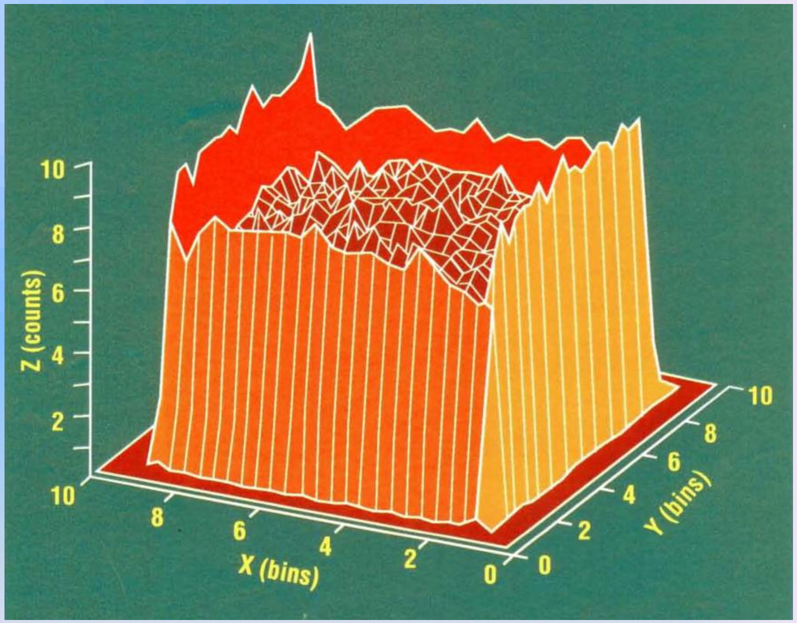

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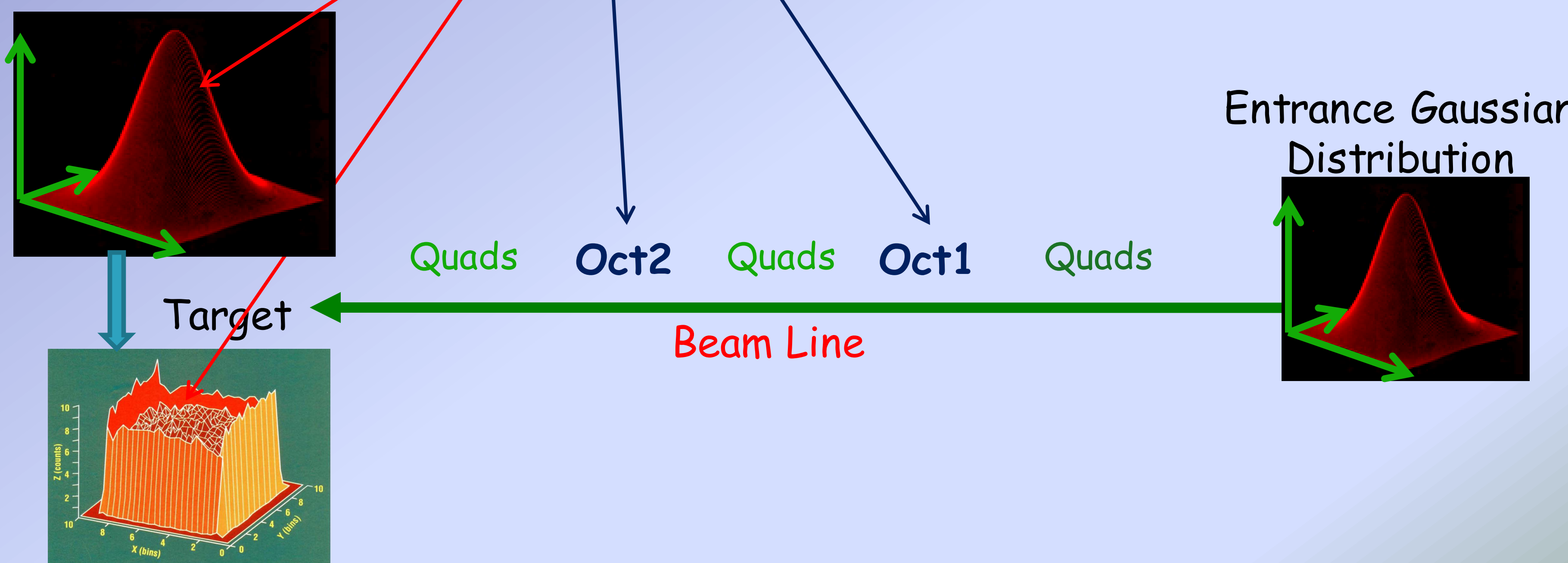
Conclusion:

In a linear beam transport line (Dipole, Quadrupoles) ;

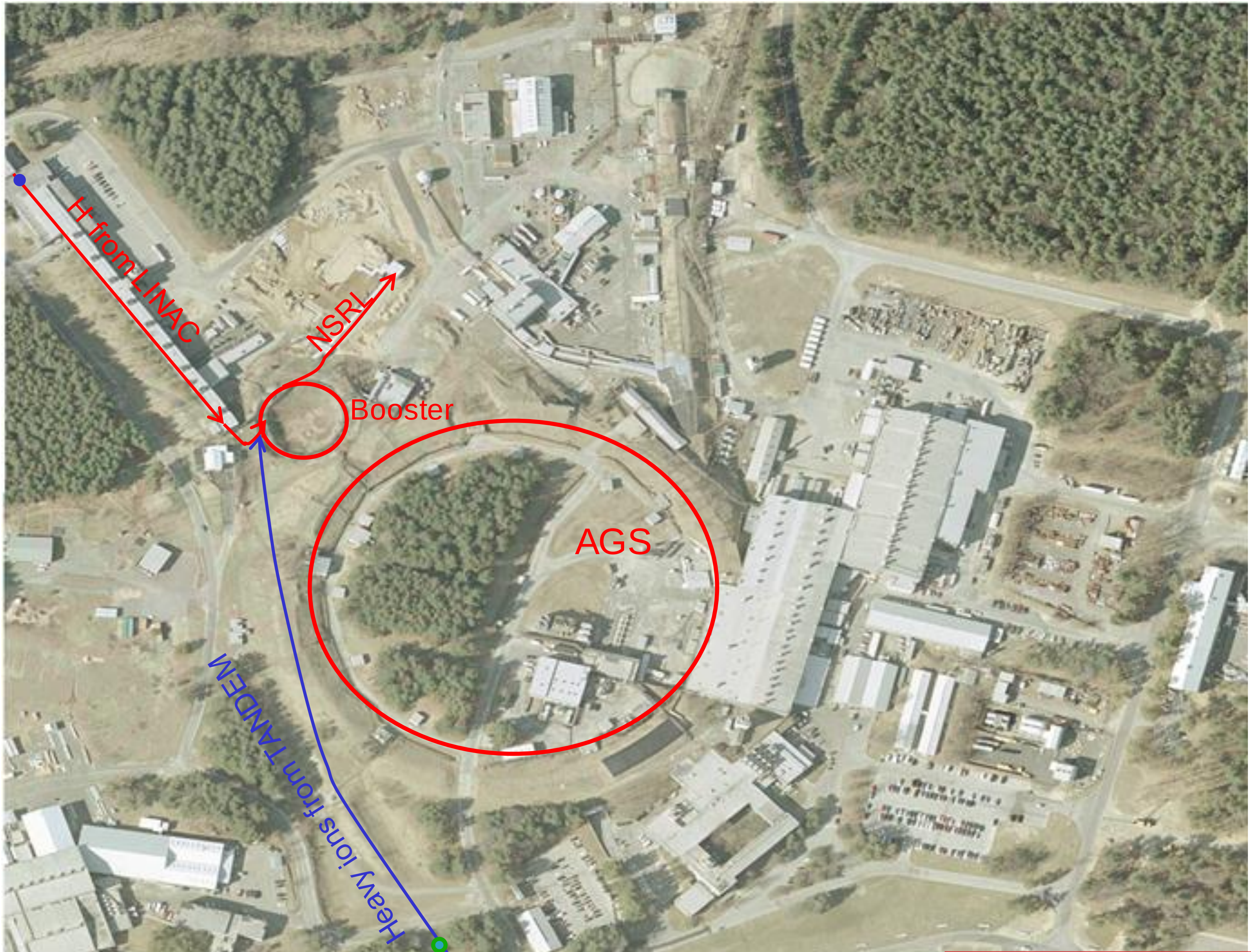
By introducing two Octupoles;

We can transform the Gaussian distribution at the Target;

Into a a beam with Uniform distribution;

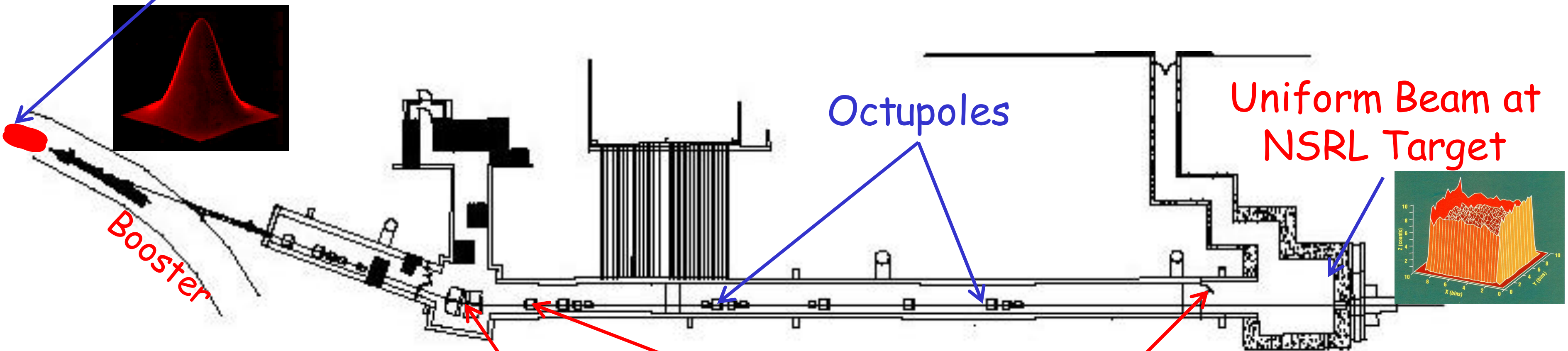


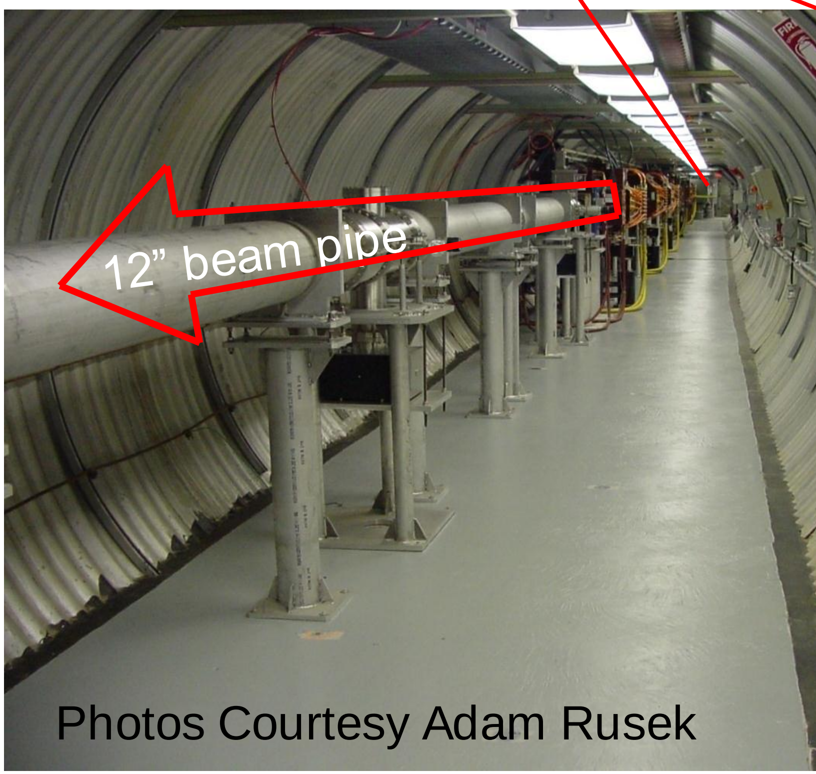
Aerial view of the NSRL facility where Uniform beam distributions are generated

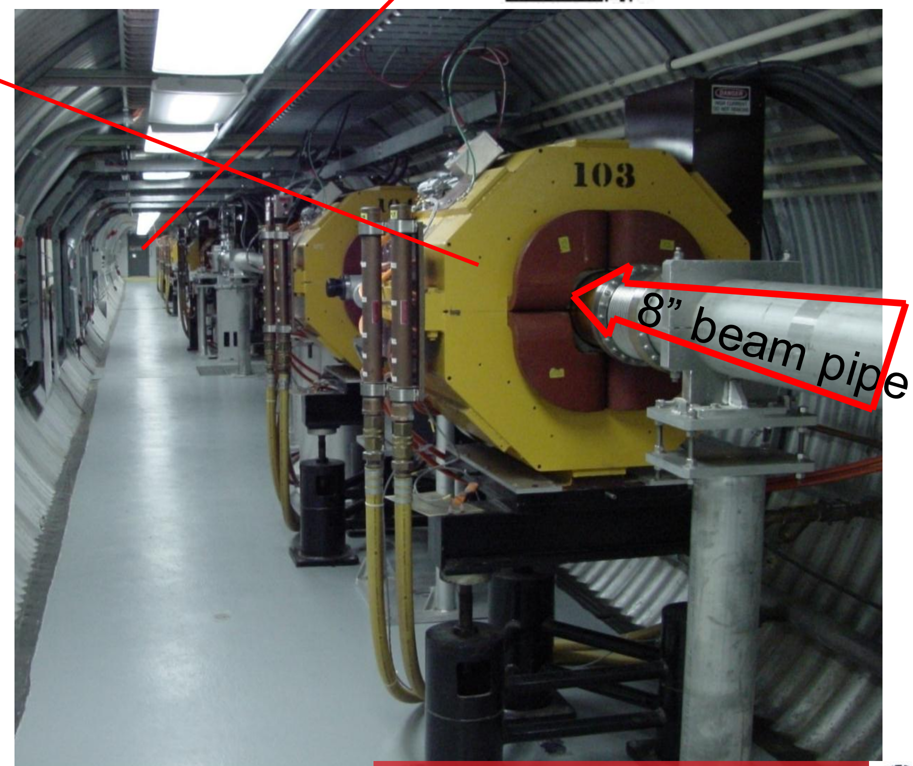


R-Line: From Booster to NSRL Facility where Uniform beam distribution are generated

Gaussian Beam at Start of R-Line

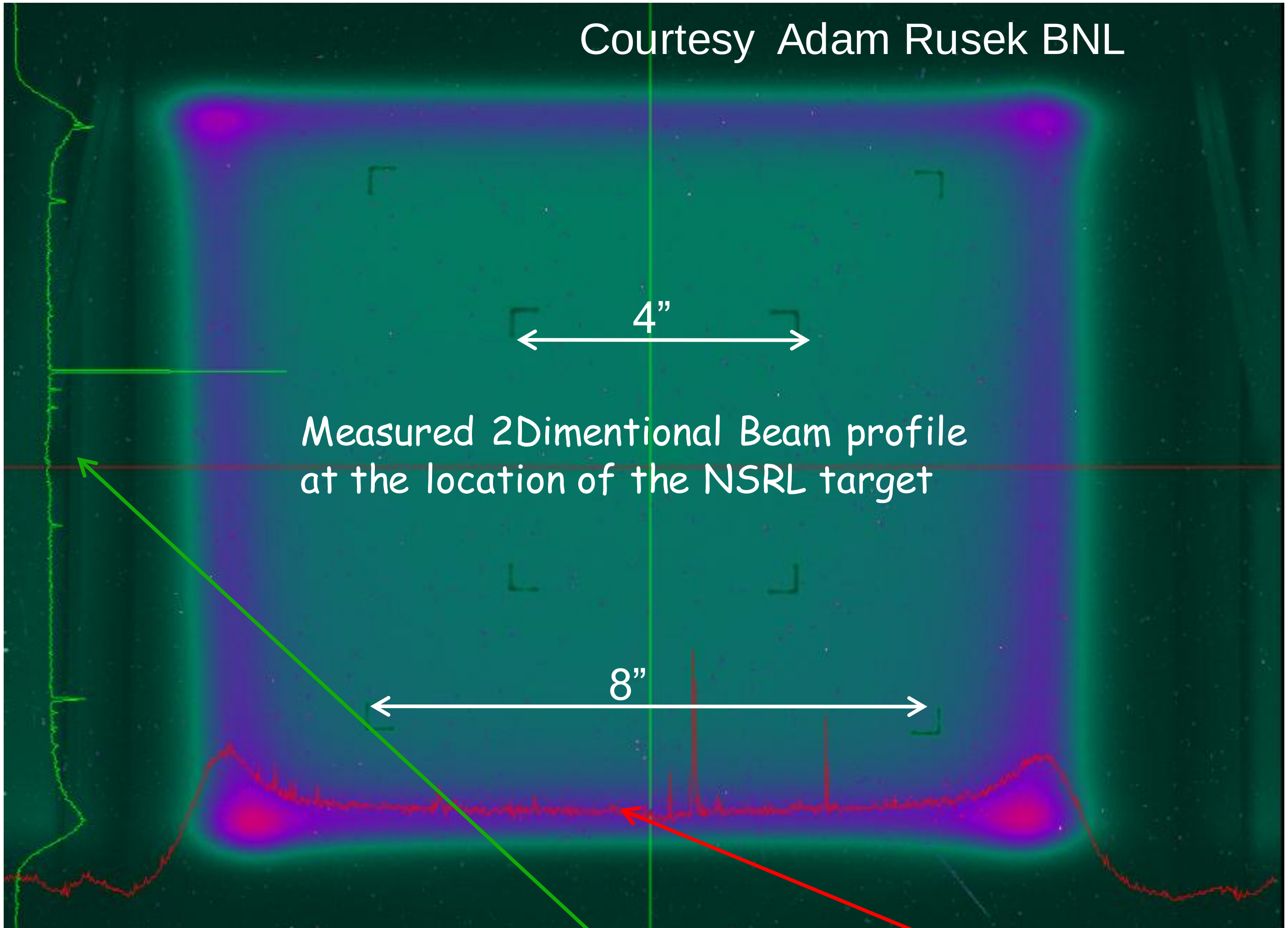




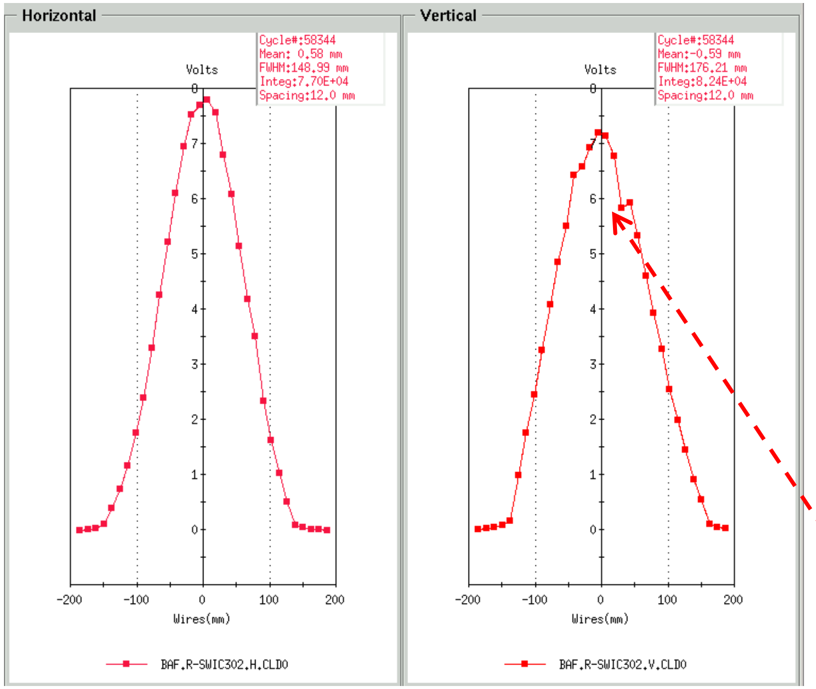


Photos Courtesy Adam Rusek

Experimental Results of Uniform beam Distributions at the target location of the NSRL facility at BNL



Measured beam profiles: Vertical (Green line) and Horizontal (Red Line) at the location of the NSRL target



Measured

Horizontal and Vertical Beam Profiles

at the location of the NSRL target

NSRL Facility at BNL

with Octupole OFF

“Bad” Harp Wire

with Octupoles ON

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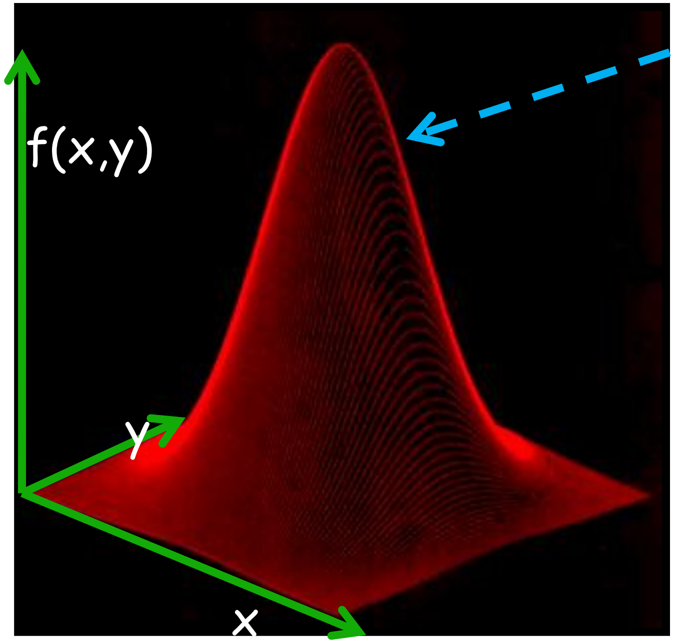
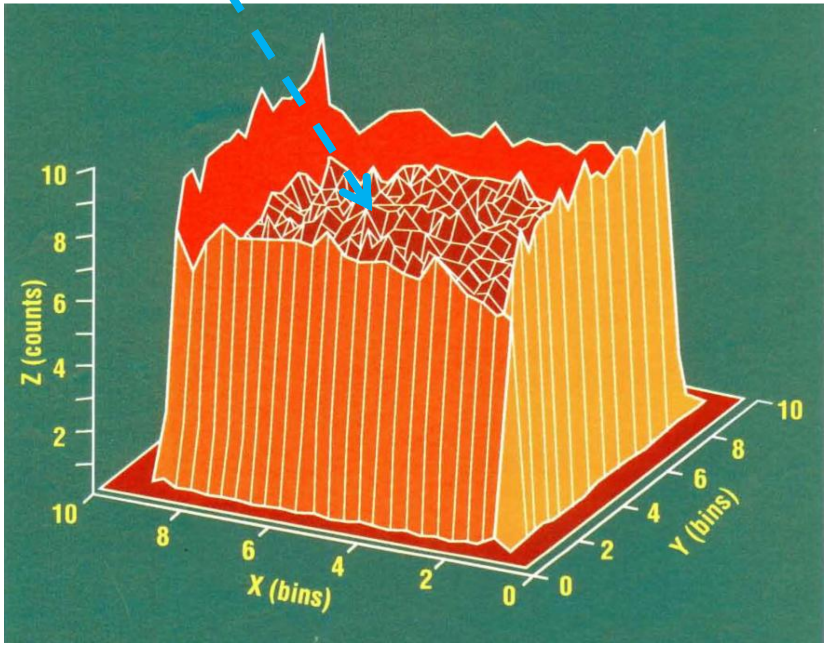
Objective of the presentation

- Present a method to transform a beam with a Gaussian distribution into beam with a Uniform distribution.





- Present an example of the method.

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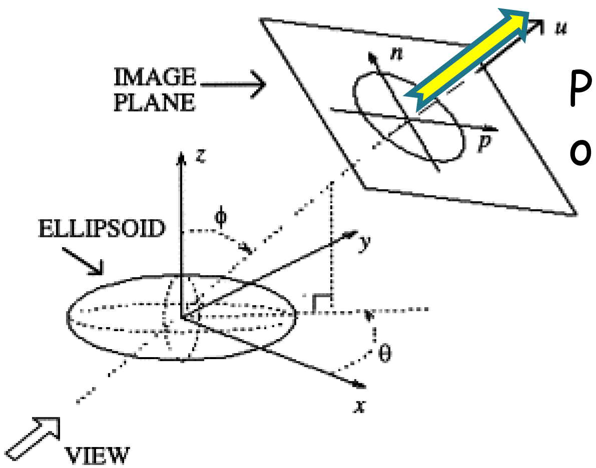
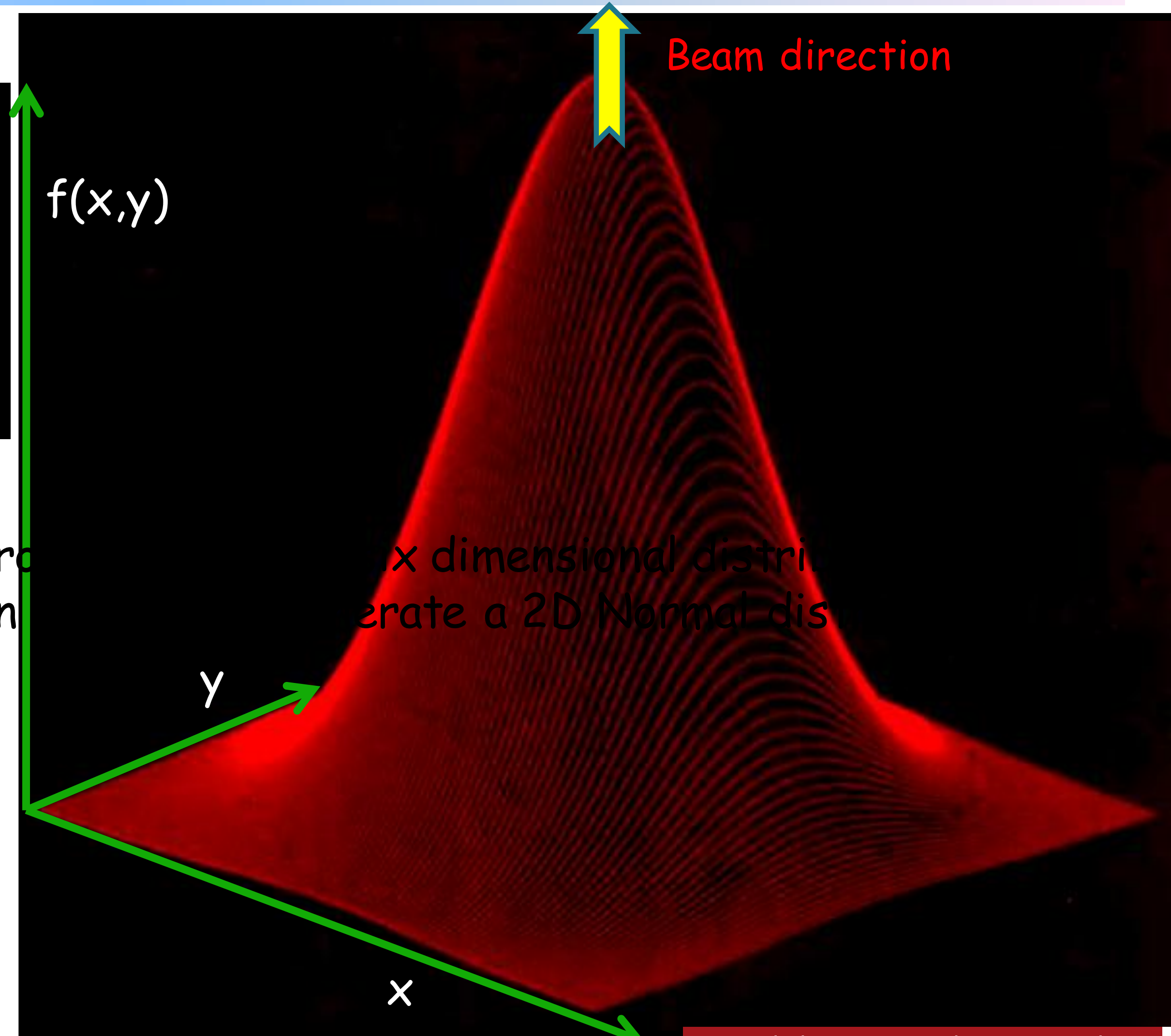
An Isometric view of a theoretical Gaussian beam distribution in the transverse (x,y) plane.

The vertical-axis corresponds to the density distribution function.

To a good approximation the particle distribution of a beam generated by an accelerator, is Gaussian in any of its six dimensions (x,x´,y,y´,$\delta\ell$, δp)





Pr… x dimensional distri…
on… erate a 2D Normal dis…

In Linear Beam Transport (Dipole Quads) The distribution of the transported beam remains Normal (next four slides)

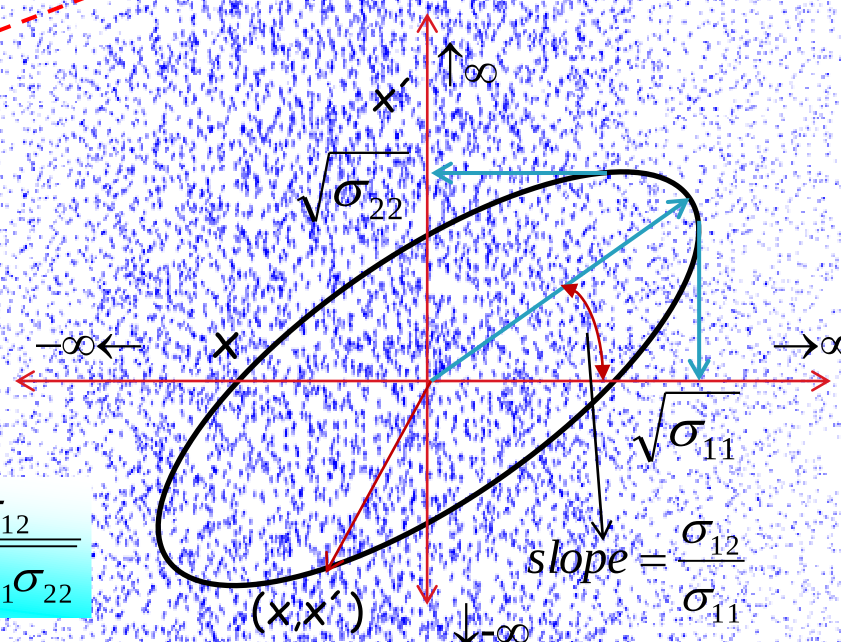
A 2-D Gaussian ("Normal") Distribution function expressed in terms of the "σ-matrix".

$$f(x,x')=\frac{1}{(2\pi)^{\frac{2}{2}}\sqrt{\det(\sigma)}}e^{-\frac{1}{2}\left(\frac{\sigma_{22}x^2}{\det(\sigma)}-\frac{2\sigma_{12}xx'}{\det(\sigma)}+\frac{\sigma_{11}x'^2}{\det(\sigma)}\right)}=\frac{1}{(2\pi)^{\frac{2}{2}}\sqrt{\det(\sigma)}}e^{-\frac{1}{2}(\tilde{X}\sigma^{-1}X)}$$

$$\tilde{X}=\{x \quad x'\}$$

$$\sigma=\begin{Bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{Bmatrix}$$

$$X=\begin{Bmatrix}x\\ x'\end{Bmatrix}$$



standard deviation(rms) in x $\equiv\sqrt{\sigma_{11}}$

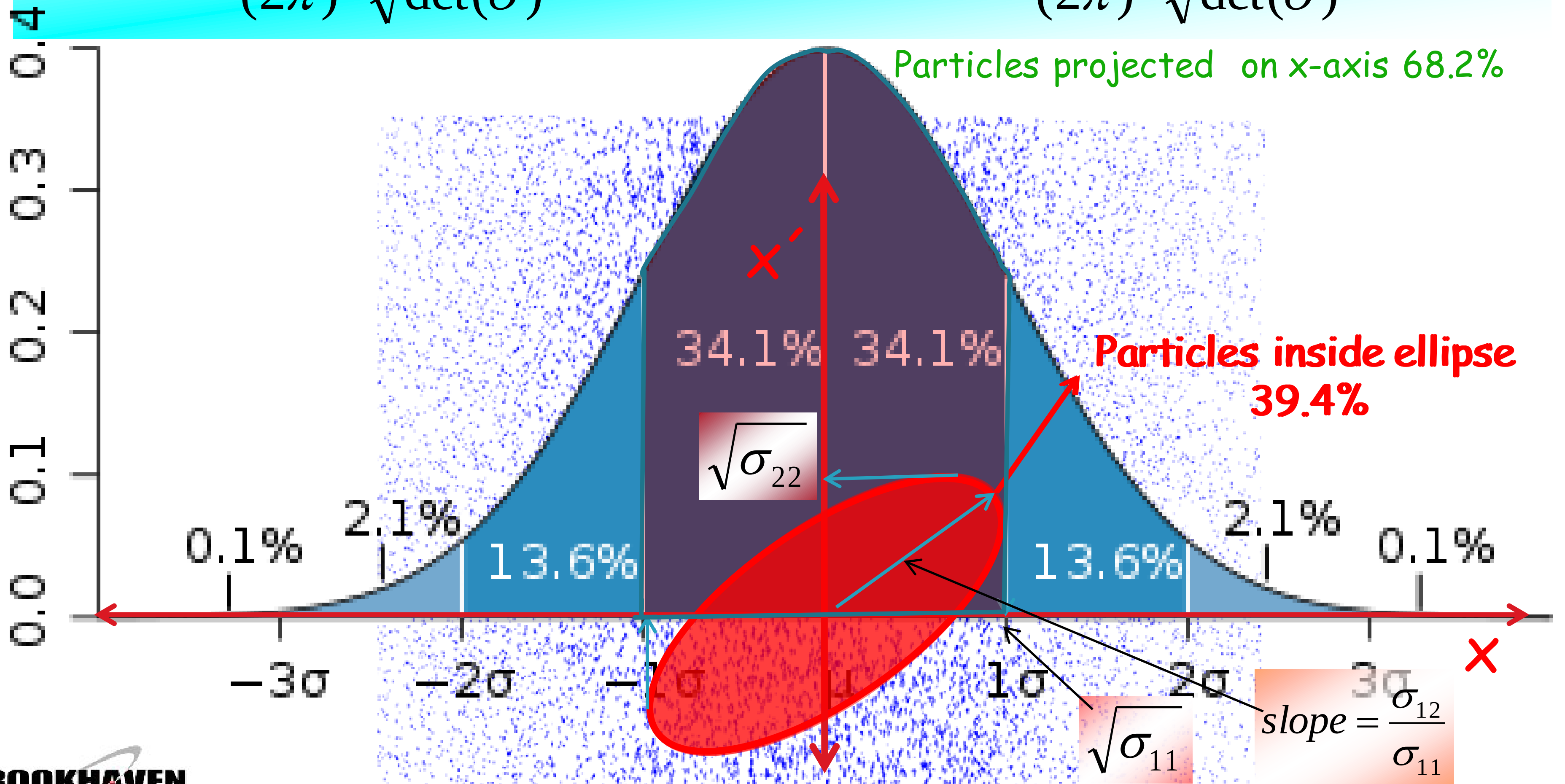
standard deviation(rms) in x' $\equiv\sqrt{\sigma_{22}}$

correlation in (x, x') $\equiv\sigma_{12}$

Correlation (r_{12}) as used in TRANSPORT => $r_{12}=\frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}}$

A 2-D Gaussian ("Normal") Distribution function expressed in terms of the "σ-matrix".

$$f(x,x') = \frac{1}{(2\pi)^{\frac{2}{2}}\sqrt{\det(\sigma)}} e^{-\frac{1}{2}\left(\frac{\sigma_{22}x^2}{\det(\sigma)} - \frac{2\sigma_{12}xx'}{\det(\sigma)} + \frac{\sigma_{11}x'^2}{\det(\sigma)}\right)} = \frac{1}{(2\pi)^{\frac{2}{2}}\sqrt{\det(\sigma)}} e^{-\frac{1}{2}(\tilde{X}\sigma^{-1}X)}$$



A charge distribution in 6D can be expressed in terms of a 6-D σ-matrix

$$X = \begin{pmatrix} x \\ x' \\ y \\ y' \\ \delta l \\ \delta p \end{pmatrix}$$

$$\tilde{X} = \{x \quad x' \quad y \quad y' \quad \delta l \quad \delta p\}$$

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} & \sigma_{35} & \sigma_{36} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} & \sigma_{45} & \sigma_{46} \\ \sigma_{51} & \sigma_{52} & \sigma_{53} & \sigma_{54} & \sigma_{55} & \sigma_{56} \\ \sigma_{61} & \sigma_{62} & \sigma_{63} & \sigma_{64} & \sigma_{65} & \sigma_{66} \end{pmatrix}$$

$$\sigma_{ij} = \sigma_{ji}$$

X => Phase space vector of a particle in beam

$$f(x, x', y, y', \delta l, \delta p) = \frac{1}{(2\pi)^{\frac{6}{2}} \sqrt{\det(\sigma)}} e^{-\frac{1}{2}(\tilde{X}\sigma^{-1}X)}$$

Also use the Notation: $\{x \quad x' \quad y \quad y' \quad dl \quad dp\} \equiv \{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6\}$

Linear Transformation of a beam with Gaussian distribution

P_{out} $x_2^{(Out)}$ $x_1^{(Out)}$

$\sigma_{out} - matrix$

Quadrupoles Transport the particle distribution from CoSys at P_{in} to CoSys at P_{out}

P_{in} $x_2^{(In)}$ $x_1^{(In)}$

$\sigma_{in} - matrix$

Beam

$$X^{(Out)} = \begin{pmatrix} x_1^{Out} \\ x_2^{Out} \\ x_3^{Out} \\ x_4^{Out} \\ x_5^{Out} \\ x_6^{Out} \end{pmatrix}$$

$$X^{(Out)} = RX^{(In)}$$

$$x_i^{(Out)} = R_{ij} x_j^{(In)}$$

$$X^{(In)} = \begin{pmatrix} x_1^{In} \\ x_2^{In} \\ x_3^{In} \\ x_4^{In} \\ x_5^{In} \\ x_6^{In} \end{pmatrix}$$

The R Matrix transforms the coordinates $X^{(In)}$ of a particle of the distribution in the CS at P_{in} to the coordinates $X^{(Out)}$ in the CS in P_{out}

$$R^T S R = S \Rightarrow |R| = 1 \;,\; R^{-1} = S^{-1} R^T S$$

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\sigma_{out} \Leftarrow \sigma_{out} = R \sigma_{in} R^T \Leftarrow \sigma_{in}$$

1. Under Linear Transformation R the beam distribution remains Gaussian
2. Transformation of a Gaussian is equivalent to Transformation of the σmatrix

Linear elements cannot change the Gaussian beam distribution

How can we change a Gaussian beam distribution
To a Non-Gaussian one?

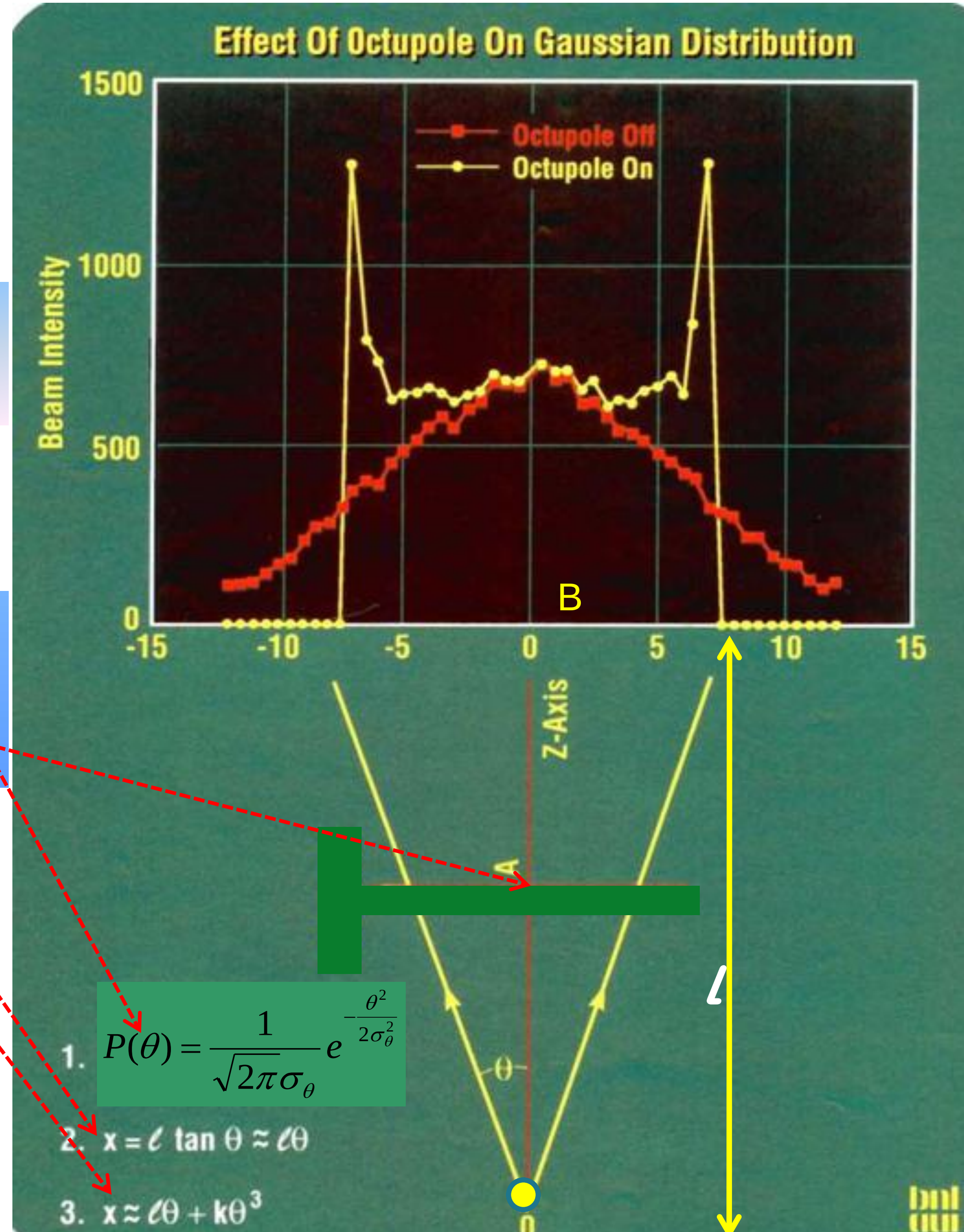
Effect of an octupole on a Gaussian beam

1. Beam is distributed Normally in angle θ as it emanates from point O.

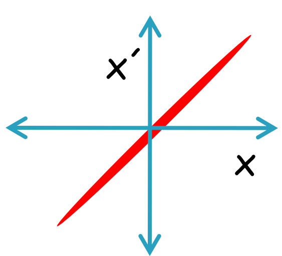
2. At $z=\ell$, $x \approx \ell\theta \Rightarrow$ the beam is distributed Normally along the x direction at a distance $z=\ell$, (red curve).

$$\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & l \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

3. Inserting an octupole at a point A, the x and θ are related by $x \approx \ell\theta + k\theta^3$ The distribution along the x, (at $z=\ell$) is modified to that of the yellow curve.



Important constraint:
At the location of the Octupole there is perfect (x,θ) correlation.



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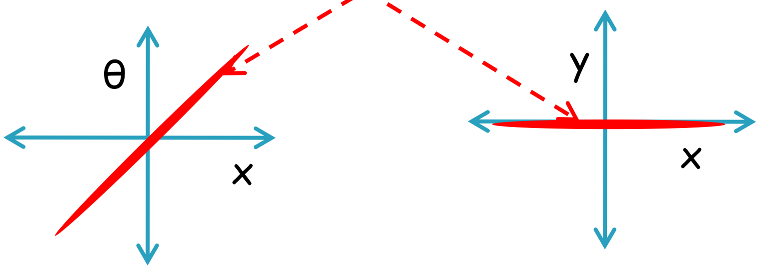
Effect of an octupole on a Gaussian beam

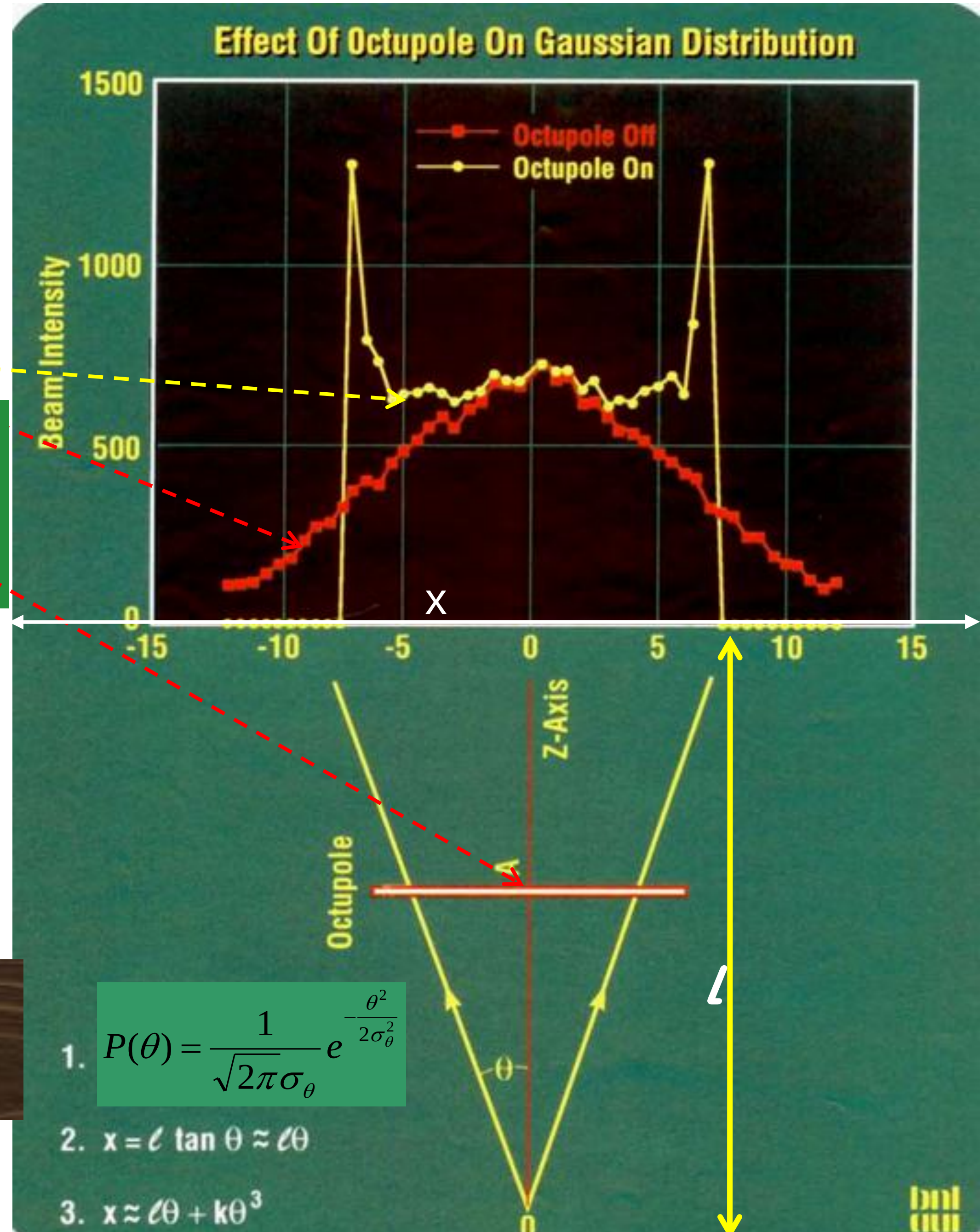
An octupole can alter the distribution of a one dimensional Gaussian beam to a beam with more Uniform distribution.

Important constraints:

a) At the location of the Octupole there is perfect (x,θ) correlation.

b) Beam is "flat"

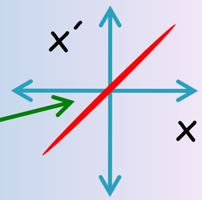




Can this simple example be modified and applied to a six dimensional realistic beam which has non-zero emittance?

Questions to be answered for generating a uniform rectangular beam in two-dimensions using “realistic” beam and octupoles.

Will the finite emittance of the beam provide the high correlation in phase space (x,x´) and (y,y´) required at the location of the octupoles, and also keep the beam within the available aperture?

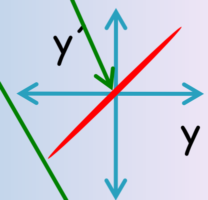


Will the finite dimensions of the “flat beam” at the location of the octupoles be affected by the aberrations introduced by the octupoles, thus affecting the beam uniformity at the target?

A“flat” beam minimizes coupling.

$$B_y(r,\theta) = -\frac{B_0}{R^3} r^3 \cos(n\theta) = \frac{B_0}{R^3}(x^3 - \overset{\square 0}{3xy^2}) \approx \frac{B_0}{R^3} x^3$$

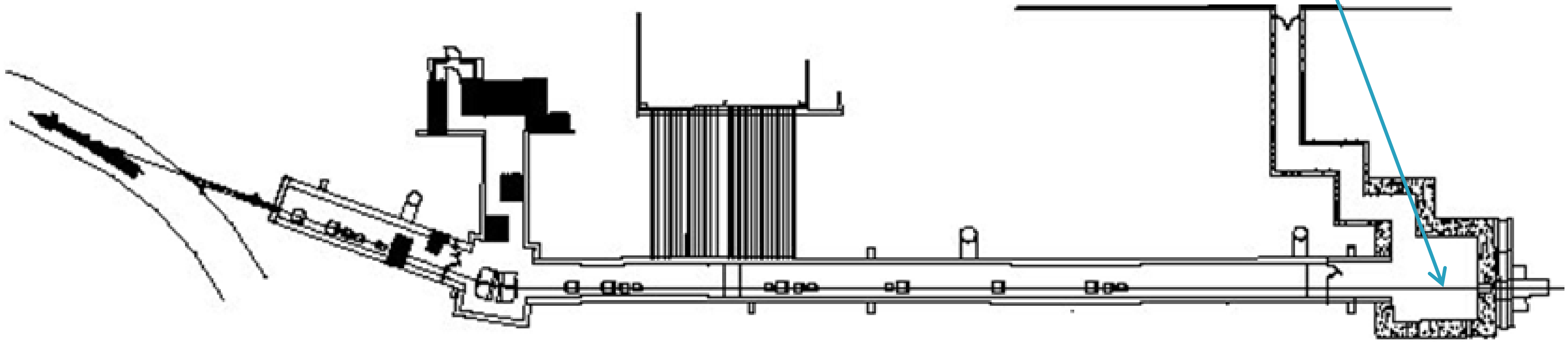
For $\sigma_y \square \sigma_x$



$$B_x(r,\theta) = -\frac{B_0}{R^3} r^3 \sin(n\theta) = \frac{B_0}{R^3}(\overset{\square 0}{3x^2 y} - y^3) \approx -\frac{B_0}{R^3} y^3$$

For $\sigma_x \square \sigma_y$

Mathematics to Calculate the beam distribution at the target



How do the equations of motion of a charged particle moving in the magnetic field of dipoles, quadrupoles, and octupoles, look like?

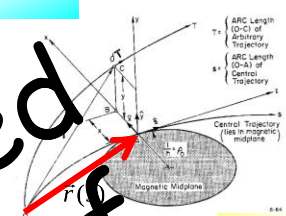
Is there a closed form solution of the exact (or approximate) equations of motion?

Do I need a computer code to generate a uniform beam distribution starting from a beam with normal distribution?

Expression of the equation of motion in the curv…

K. L. Brown SLAC-PUB 3381

$$\frac{d\vec{\mathbf{p}}}{dt} = e(\vec{\mathbf{v}}\times\vec{\mathbf{B}})$$



$$h = \frac{1}{\rho}$$

These equations are exact

The x-componen…

$$\{x'' - h(1+hx) - \ldots \quad \ldots (y' B_s - (1+hx) B_y)$$

The y-component

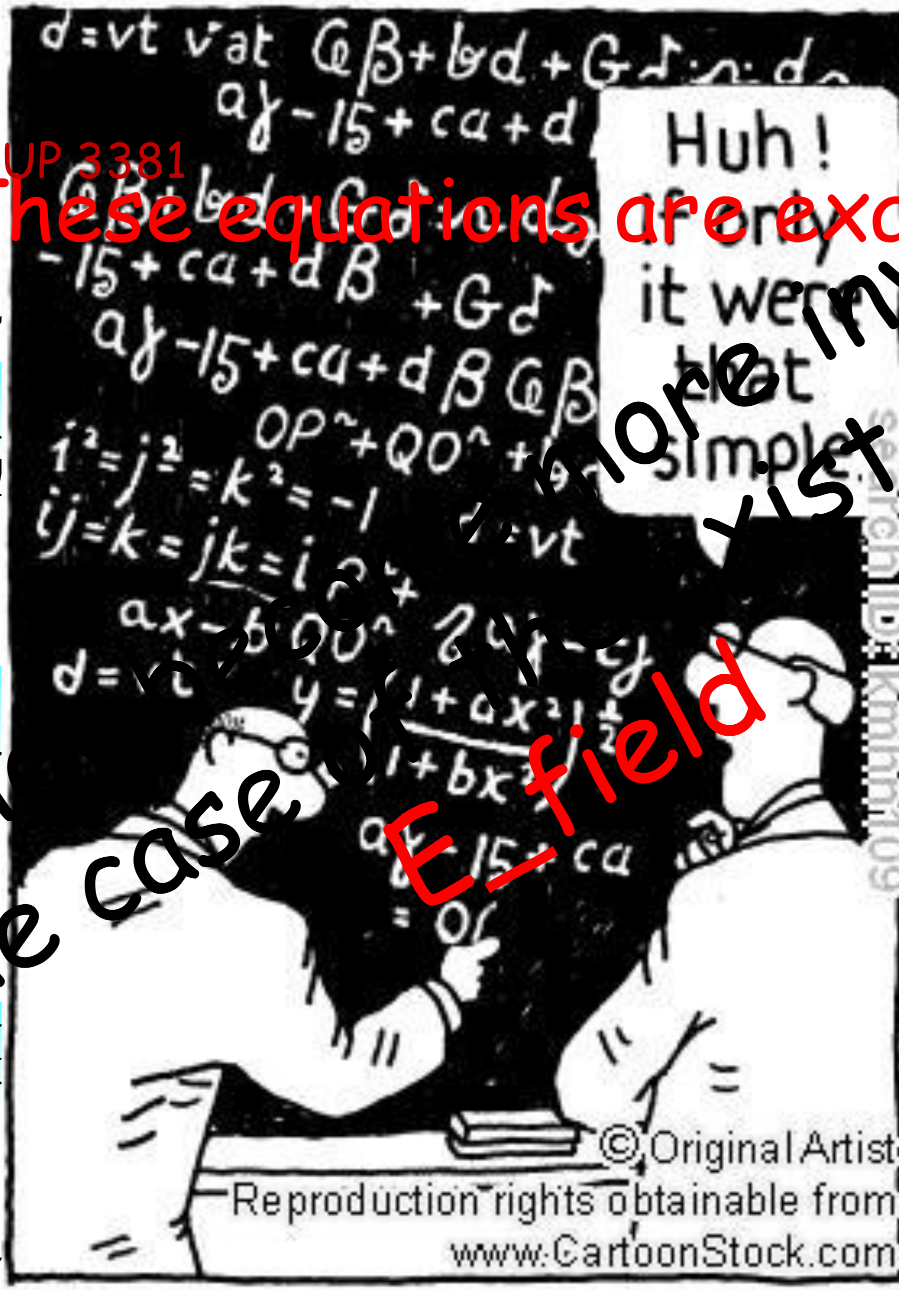
$$\{y'' - \frac{y'}{T'^2}[x'x'' + \ldots \quad \ldots (1+hx) B_x - x' B_s]$$

The s-componen…

$$\{(2hx' + h'x) - \ldots \quad \ldots \} = \frac{e}{p} T'(x' B_y - y' B_x)$$

$$(T')^2 = (x')^2 + (y')^2 + (\ldots \quad \ldots x + (x')^2 + (y')^2]^{1/2}$$

But th… more involved in the case of the existence of E_field



Fields expansion to 3rd order in coordinates: B_x, B_y, B_s

$$B_x = \frac{p_0}{e}K_1(s)y + \frac{p_0}{e}K_2(s)xy + \frac{p_0}{e}K_3(s)\left(\frac{x^2y}{2} - \frac{y^3}{3!}\right) +$$

$$\frac{p_0}{e}\left[h^2K_1(s) - hK_2(s)\right]\frac{y^3}{3!} + \frac{p_0}{e}\left[h'\frac{\partial B_y(0,0,s)}{\partial s} + 2h\frac{\partial^2 B_y(0,0,s)}{\partial s^2} - \frac{\partial^2 K_1(s)}{\partial s^2}\right]\frac{y^3}{3!}$$

$$B_y = B_y(0,0,s) + \frac{p_0}{e}K_1(s)x + \frac{p_0}{e}K_2(s)\left(\frac{x^2}{2} - \frac{y^2}{2}\right) + \frac{p_0}{e}K_3(s)\left(\frac{x^3}{3!} - \frac{xy^2}{2!}\right) -$$

Dipole Quadrupole Sextupole Octupole

$$\frac{p_0}{e}\left[\frac{\partial^2 B_y(0,0,s)}{\partial s^2} + hK_1(s)\right]\frac{y^2}{2} +$$

The MAD Program CERN/SL/92-?? (AP)
B_2 expression Eq. 2.3 page 3

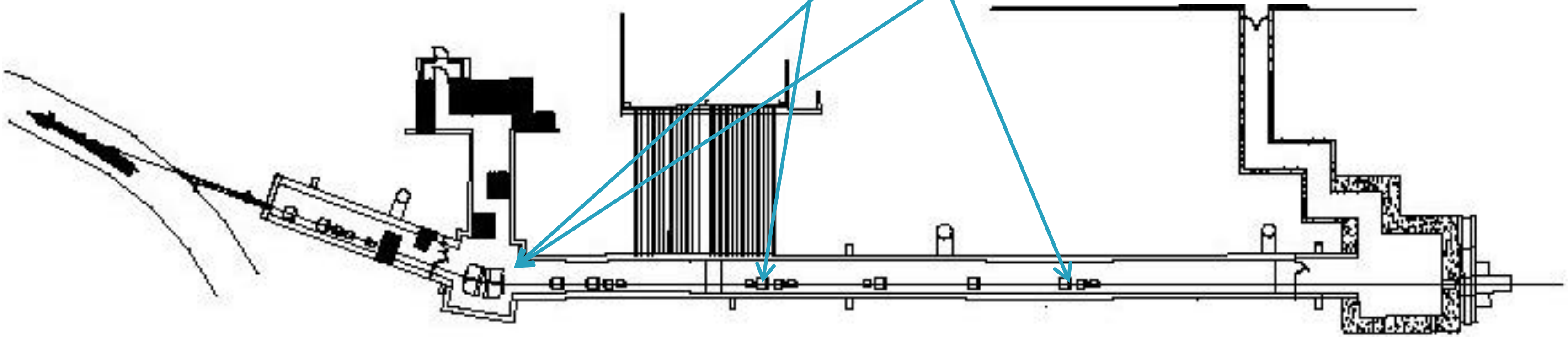
$$+\frac{p_0}{e}\left[-\frac{\partial^2 K_1(s)}{\partial s^2} + 2h\frac{\partial^2 B_y(0,0,s)}{\partial s^2} + h'\frac{\partial B_y(0,0,s)}{\partial s} - hK_2(s) + h^2K_1(s)\right]\frac{xy^2}{2!}$$

$$B_s = \frac{p_0}{e}\frac{\partial B_y(0,0,s)}{\partial s}y + \frac{p_0}{e}\left(\frac{\partial K_1(s)}{\partial s} - h\frac{\partial B_y(0,0,s)}{\partial s}\right)xy + \frac{p_0}{e}\left(\frac{\partial K_2(s)}{\partial s} - h\frac{\partial K_1(s)}{\partial s}\right)x^2y$$

$$-\frac{p_0}{e}\left(\frac{\partial^3 B_y(0,0,s)}{\partial s^3} + \frac{\partial K_2(s)}{\partial s} + h'\frac{p_0}{e}K_1(s) + h\frac{\partial K_1(s)}{\partial s}\right)y^3$$

In an attempt to simplify the equations of motion we start eliminating multipoles.

If we assume that the Gaussian beam starts at the exit of the last dipole of the line, we may eliminate the dipole field in the equations of motion and use quadrupoles and octupoles only.



The equations of motion are Simplified !!!! to:

The x-component

$$\{x'' - x'^2 x'' - x'y'y''\} \approx -K_1(s)x - K_3(s)\left(\frac{x^3}{3!} - \frac{xy^2}{2!}\right) + \left(\frac{\partial K_1(s)}{\partial s}\right)xyy' + \left[\frac{\partial^2 K_1(s)}{\partial s^2}\right]\frac{xy^2}{2!}$$

Even if we drop the terms of the fringe field.....

If we assume□0

$$\{x'' - x'^2 x'' - x'y'y''\} \approx -K_1(s)x - K_3(s)\left(\frac{x^3}{3!} - \frac{xy^2}{2!}\right)$$

To third order with Quadrupoles and Octupoles only

$$x'' \approx -K_1(s)x - K_3(s)\left(\frac{x^3}{3!} - \frac{xy^2}{2!}\right)$$

This approximation Not correct!!!

$K_3(s)$ cannot be treated as a perturbation

The y-component

$$y'' - x'y'x'' - y'y'y'' = K_1(s)y + \frac{1}{2}K_1(s)(yx'^2 + yy'^2) + K_3(s)\left(\frac{x^2y}{2} - \frac{y^3}{3!}\right) - \left(\frac{\partial^2 K_1(s)}{\partial s^2}\right)\frac{y^3}{3!} - \left(\frac{\partial K_1(s)}{\partial s}\right)xyx'$$

Even if we drop the terms of the fringe field....

assume □0

$$y'' - x'y'x'' - y'y'y'' = K_1(s)y + \frac{1}{2}K_1(s)(yx'^2 + yy'^2) + K_3(s)\left(\frac{x^2y}{2} - \frac{y^3}{3!}\right)$$

To third order with Quadrupoles and Octupoles only

$$y'' = K_1(s)y + \frac{1}{2}K_1(s)(yx'^2 + yy'^2) + K_3(s)\left(\frac{x^2y}{2} - \frac{y^3}{3!}\right)$$

This approximation is Not correct!!!

$K_3(s)$ cannot be treated as a perturbation

$1/\rho = h = 0$

The s-component

$$\{(2hx' + h'x) - \frac{(1+hx)}{T'^2}[x'x'' + y'y'' + (1+hx)(hx' + h'x)]\} = \frac{e}{p}T'(x'B_y - y'B_x)$$

$$[x'x'' + y'y''] = K_1(s)\left(y'y - x'x\right)$$

To third order with Quadrupoles and Octupoles only

How can one solve these equations?

$$\{x'' - x'^2 x'' - x'y'y''\} \approx -K_1(s)x - K_3(s)\left(\frac{x^3}{3!} - \frac{xy^2}{2!}\right)$$ x-comp

$$y'' - x'y'x'' - y'y'y'' \approx K_1(s)y + \frac{1}{2}K_1(s)(yx'^2 + yy'^2) + K_3(s)\left(\frac{x^2 y}{2} - \frac{y^3}{3!}\right)$$ y-comp

$$[x'x'' + y'y''] \approx K_1(s)\left(y'y - x'x\right)$$ s-comp

Quadrupoles and Octupoles only; No fringe field; Correct to 3rd order.

Can I solve these approximate equations in a closed form to calculate the beam distribution at the target?

Answer: NO

I solve the "exact equations"; But Numerically using a computer

Computer codes used: TRANSPORT , or RAYTRACE

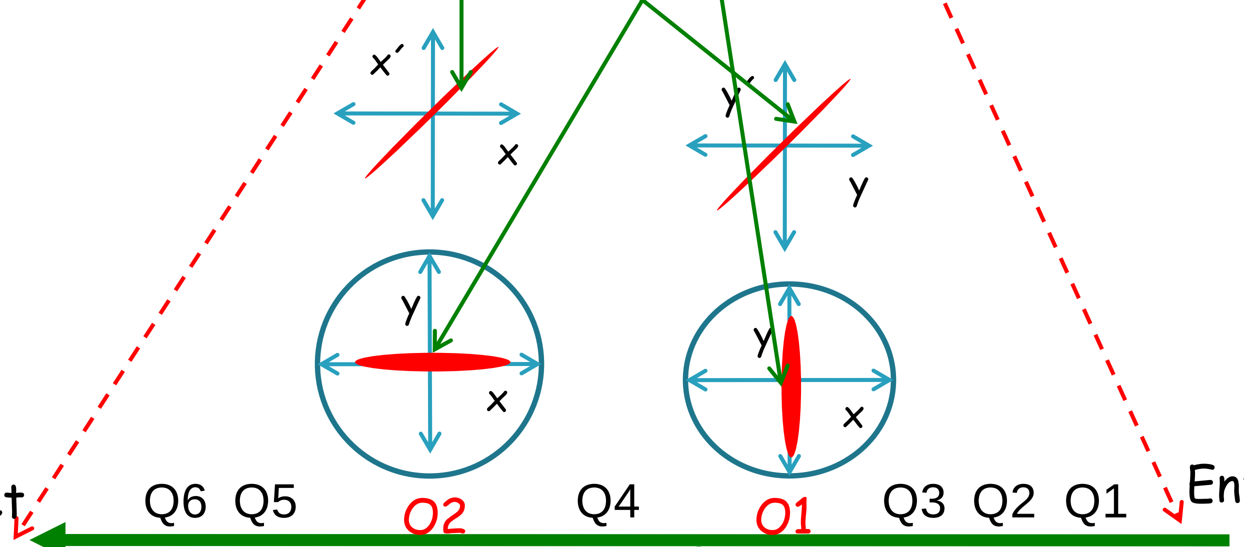
I use the TRANSPORT or RAYTRACE computer code to calculate the beam distribution at the target

Procedure to generate a uniform beam distribution

Procedure to Generate uniform beam distribution at target. 1st Order Optics

Beam distribution at the entrance of beam line is assumed to be Gaussian

- First Order Optics:
  - Should satisfy the required beam constraints at the location of the Octupoles.
    - High correlation in (x,x') and "flat beam" in y-direction at the location of one Octupole
    - High correlation in (y,y') and "flat beam" in x-direction at the location of other Octupole
  - Beam size at the target should be adjusted by the first order optics.



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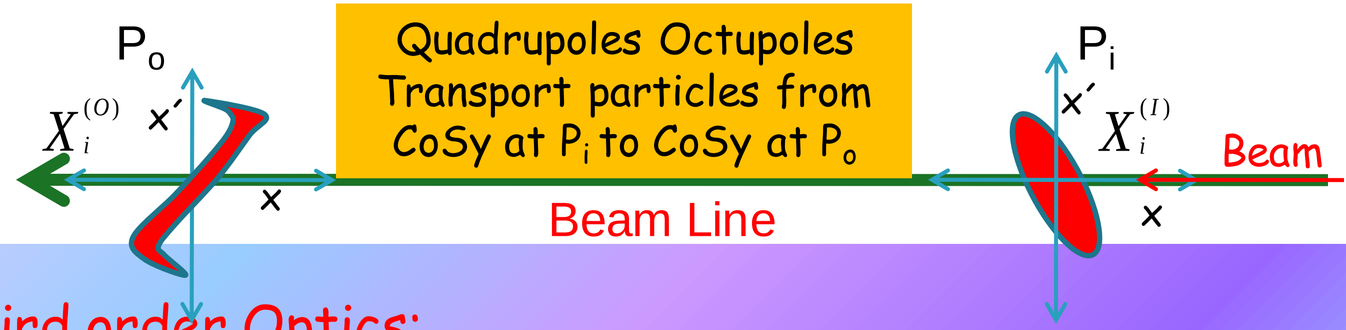
Procedure to Generate uniform beam distribution at target.
3^{rd} Order Optics



- Third order Optics:
  - With Octupoles ON, Calculate the first order (R_{ij}) , second order (T_{ijk}) , and third order (W_{ijkl}) , aberrations coefficients of the beam line.

  - Use the transformation:

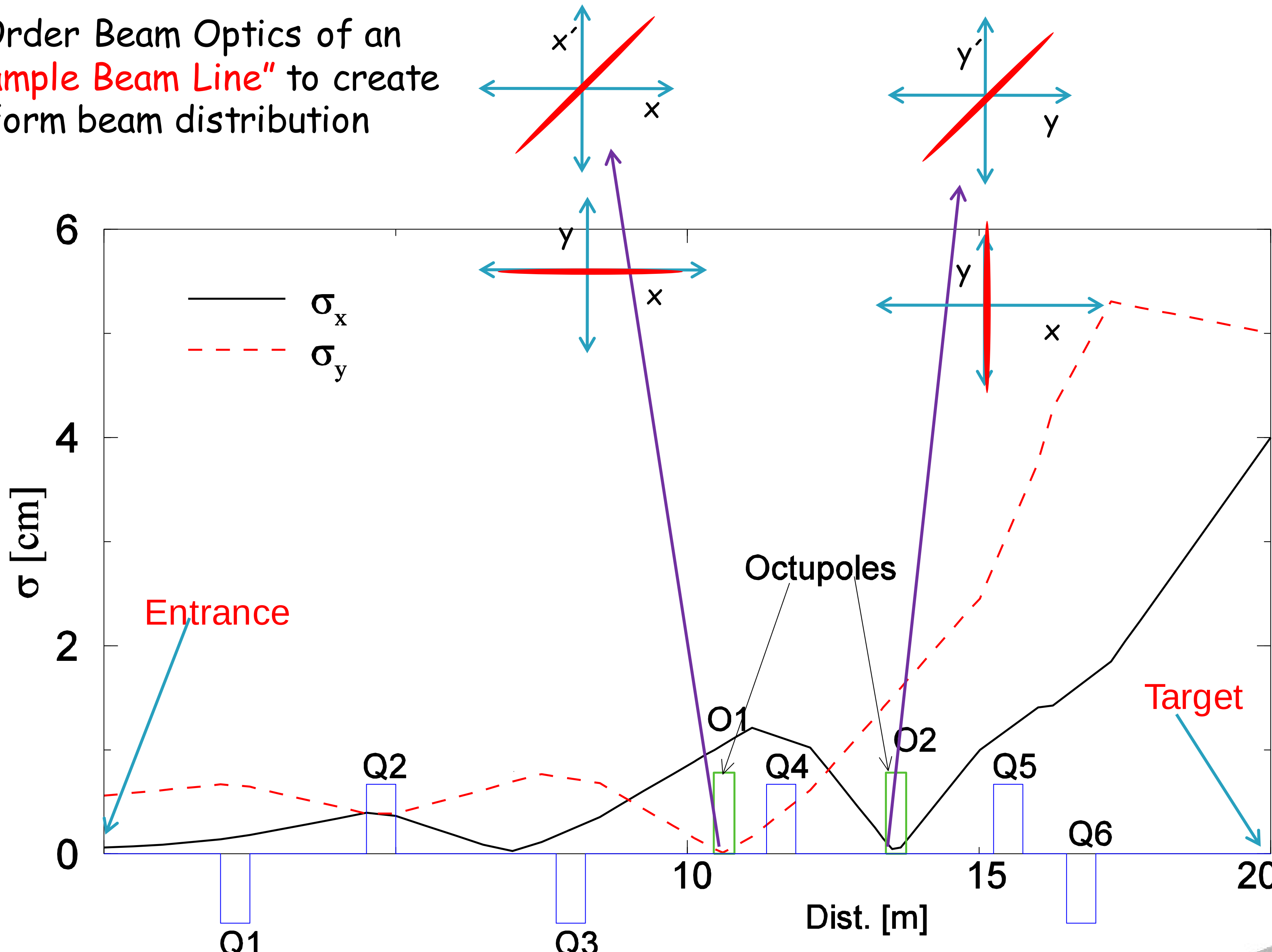
$$x_i^{(O)} = R_{ij}\, x_j^{(I)} + T_{ijk}\, x_j^{(I)} x_k^{(I)} + W_{ijkl}\, x_j^{(I)} x_k^{(I)} x_l^{(I)} \qquad (1)$$

of entrance beam coordinates $x^{(I)}$ to the exit beam coordinates $x^{(O)}$

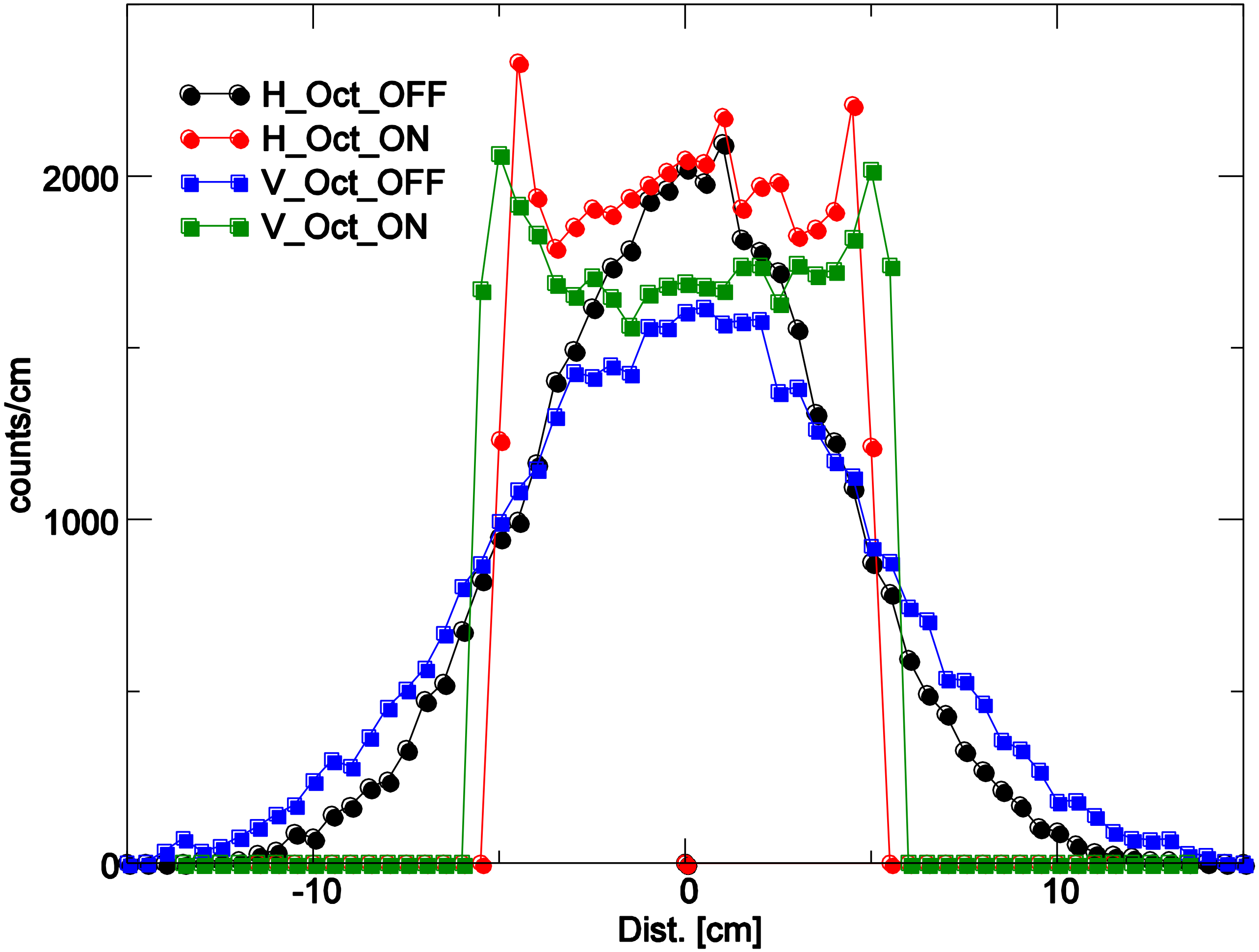
  - Perform Monte Carlo calculations using eq. (1) to calculate the particle distribution at the target.

Reminder: The "sampling" beam distribution at the entrance is Gaussian:

1st Order Beam Optics of an "Example Beam Line" to create Uniform beam distribution



Horizontal (Black/Red circles) and Vertical (Blue/Green squares) Beam Profiles with Octupoles OFF(Black/Blue) and Octupoles ON (Red/Green) for the "Example Beam Line"



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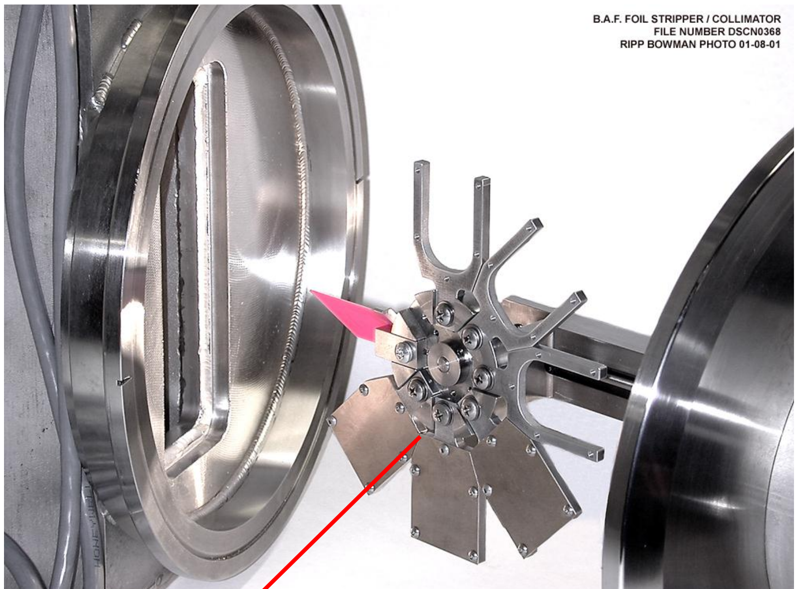
Same method of generating uniform beams but Alternative Mathematical approach

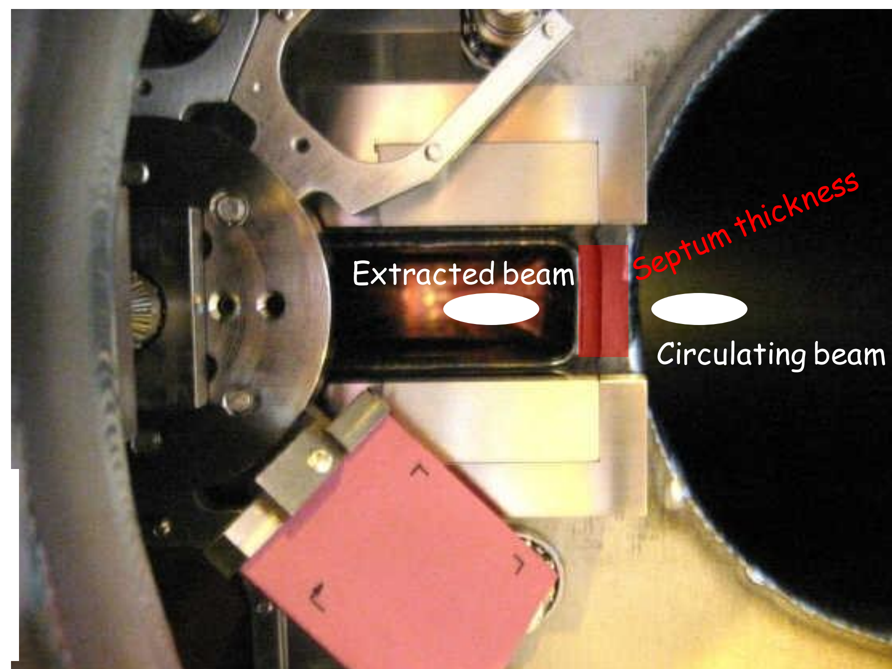
Calculations of nonlinear envelopes in beam expanders
F. Meot Physical Review Special Topics **3 10351 (2000)**

Comments

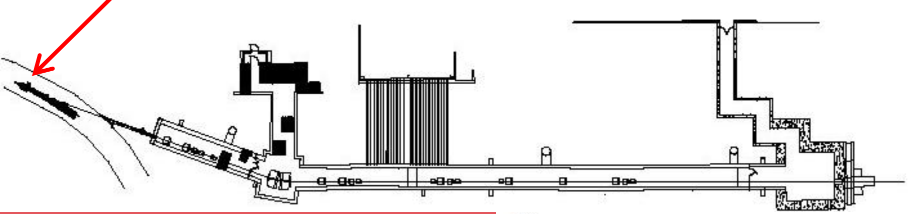
It is very important that the beam distribution at the beginning of the line is "Gaussian". How do we achieve a beam with Gaussian distribution from a slowly extracted beam at the entrance of NSRL?



0:	0	Blank
1:	0	.020 inch Cu foil
2:	1	.002 inch Cu foil
3:	0	.0007 inch Tungsten wire
4:	0	.005 inch Tungsten wire
5:	0	.002 inch Cu wire
6:	0	.020 inch Cu wire
7:	0	Flag
	1	Geneva Switch



Location of foil at the entrance of extraction septum.

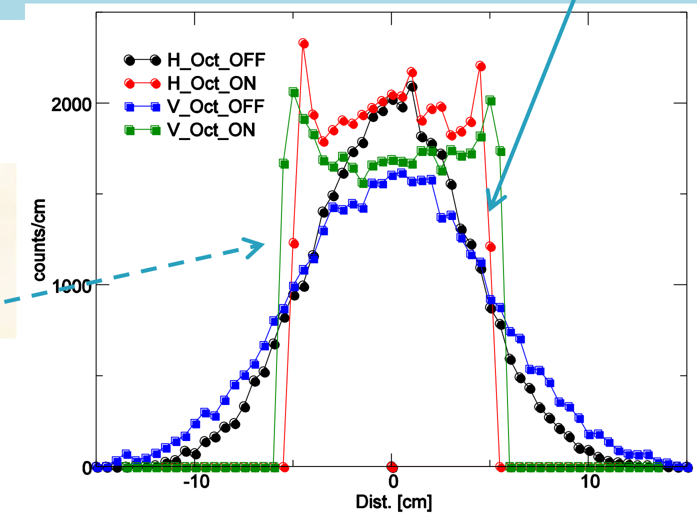


Comments

The beam emittance has an effect on the "Sharpness" of the beam distribution

Beam with Low emittance

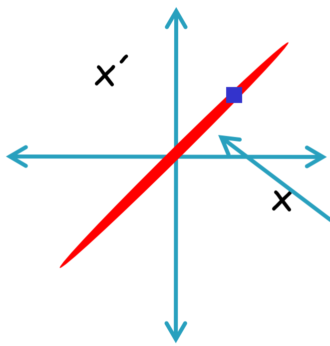
If (x,x′) highly correlated
Δx′ is small, then the kick from
the octupole will not spread the
beam at the target. "Sharp fall"

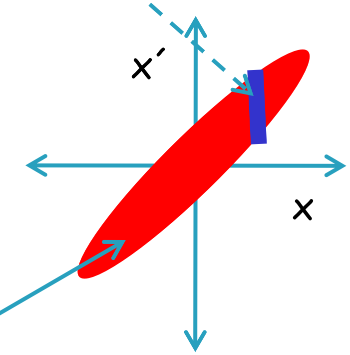


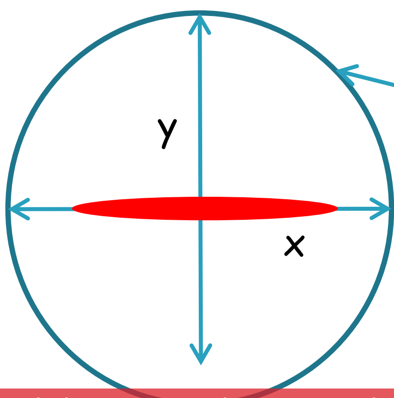
Beam profile at the location of the target

Beam with High emittance

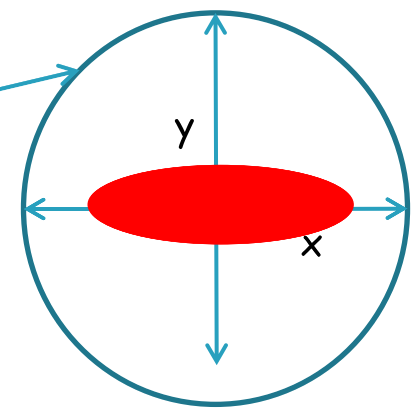
(x,x′) less correlated
Δx′ is large.
The kick from the octupole
will spread the beam at the target.
"Not Sharp fall"



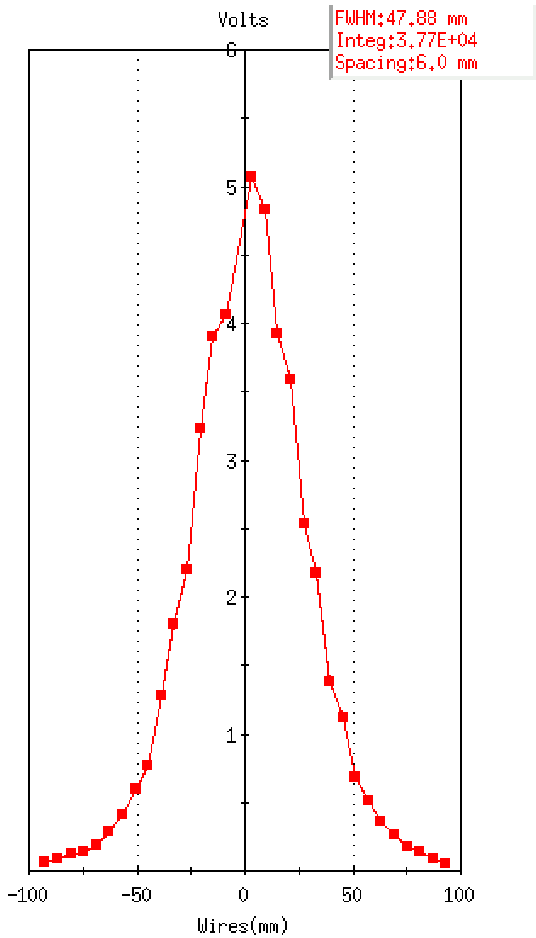




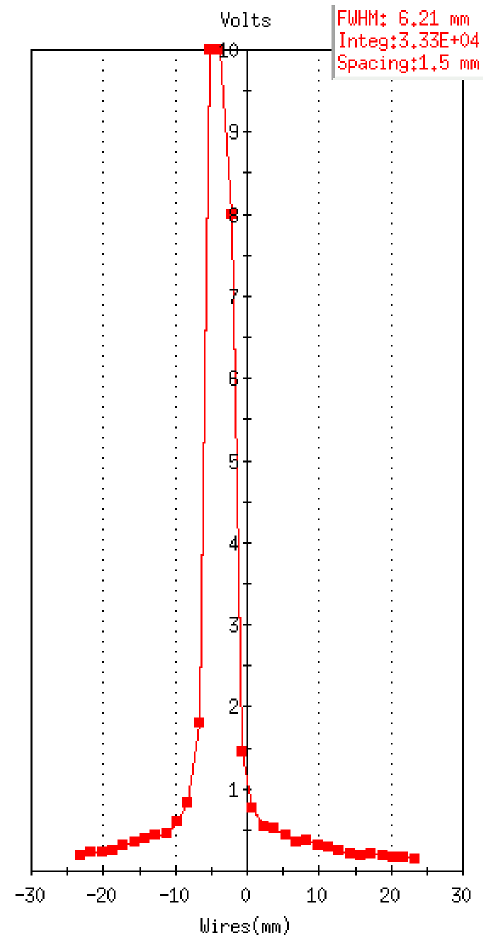
Beam at the location of the octupole



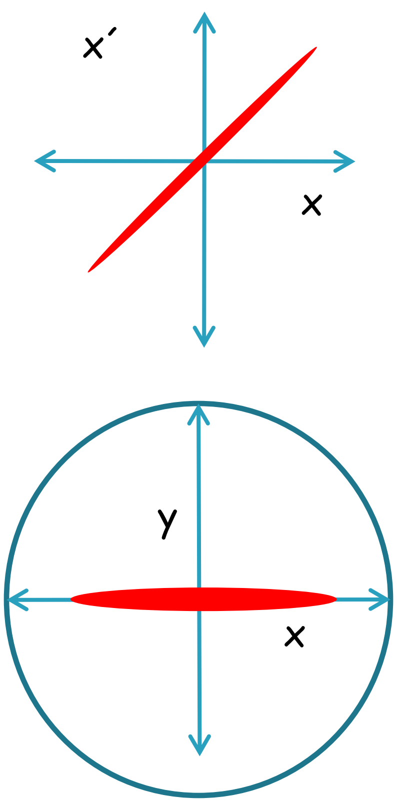
Instrumentation for beam diagnostics.
Beam profiles in front of the "Horizontal Octupole"



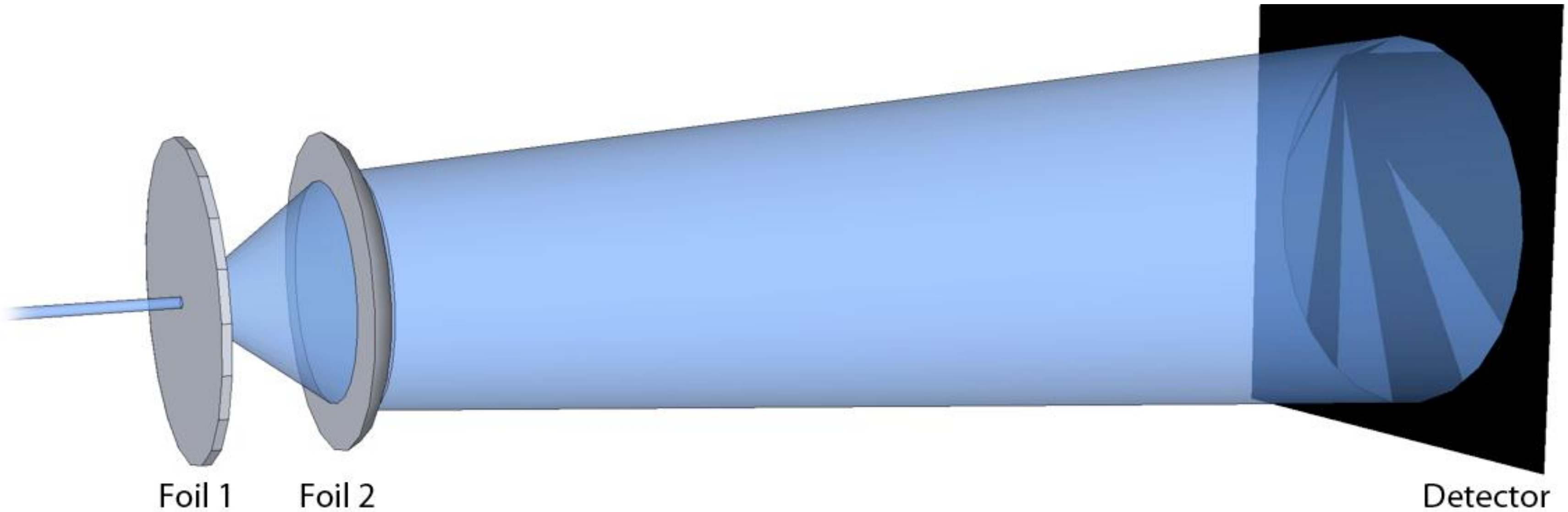
Horizontal profile



Vertical profile



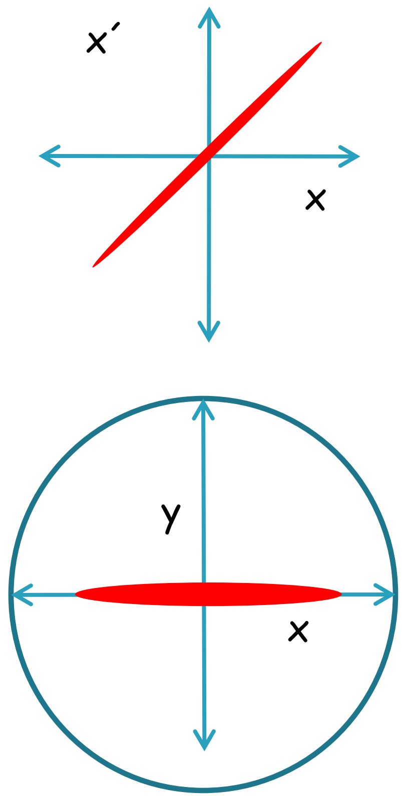
Study the Effect of the vacuum windows, on the beam. How does it affect the beam's uniformity and the "sharp fall off" of the beam?



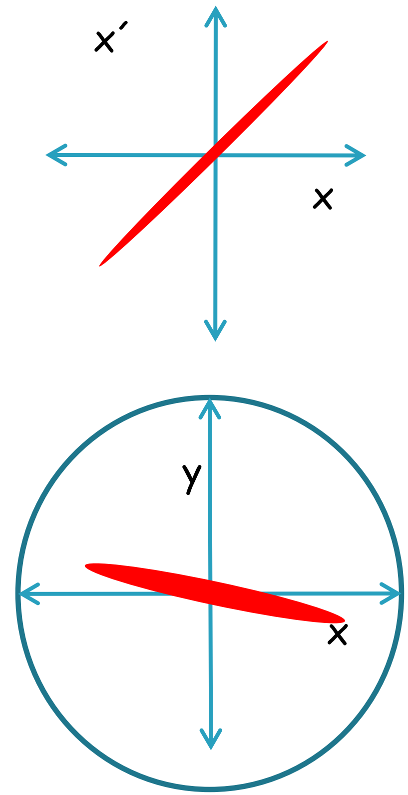
Monte Carlo simulation of particle beam flattening Using a dual-scattering-foil technique
E.Detsi

deqsolver.com/download/montecarloSim.pdf

Effect of linear coupling of the beam on the beam uniformity



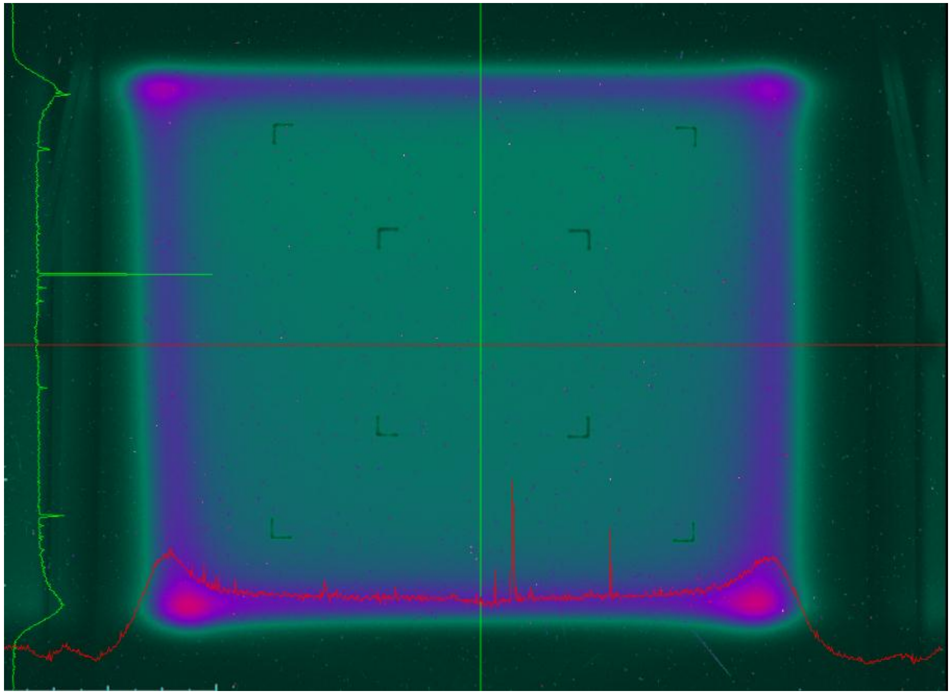
Beam with No coupling



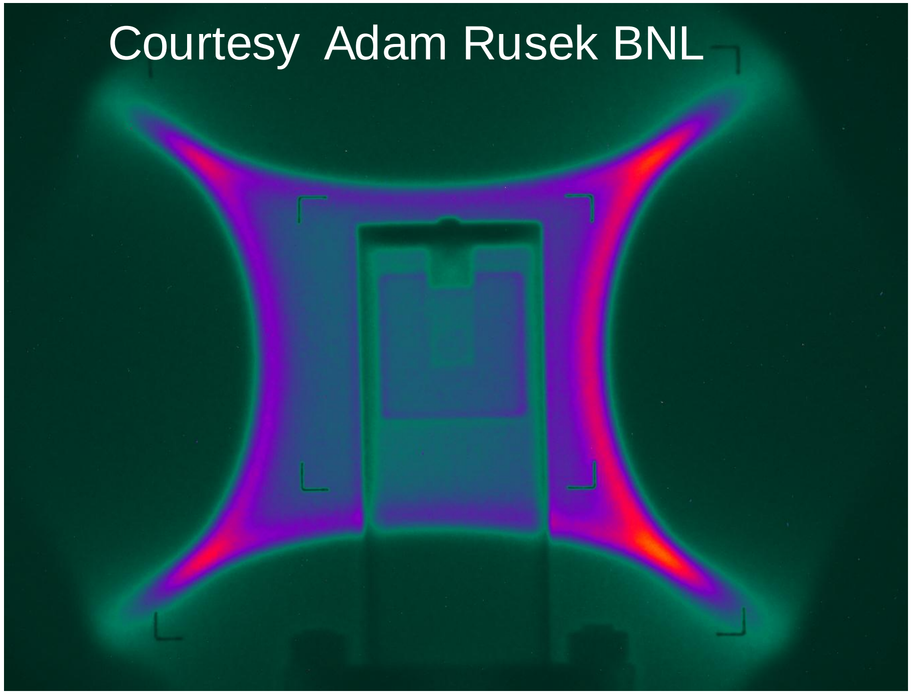
Linearly Coupled beam

Effect of Higher order coupling of the beam on the beam uniformity

Under Normal operating conditions the octupoles introduce “minimal coupling” to affect the beam uniformity and sharpness of the “beam’s fall off”



2D beam profile at target with the required strength of octupoles



2D beam at target with strong octupoles

It is Possible to identify the octupole aberrations from the distribution at the target

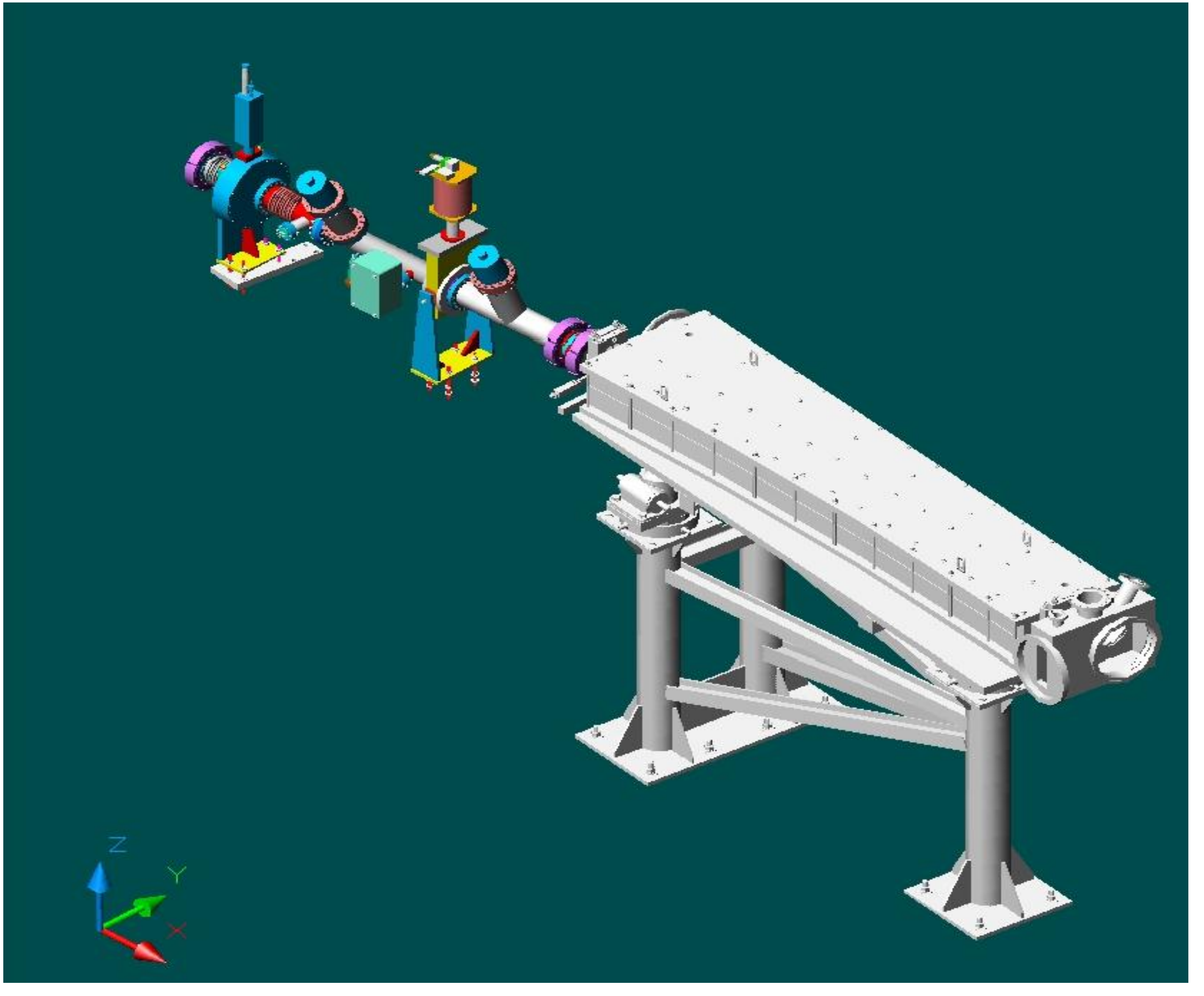
THE END

THANK YOU

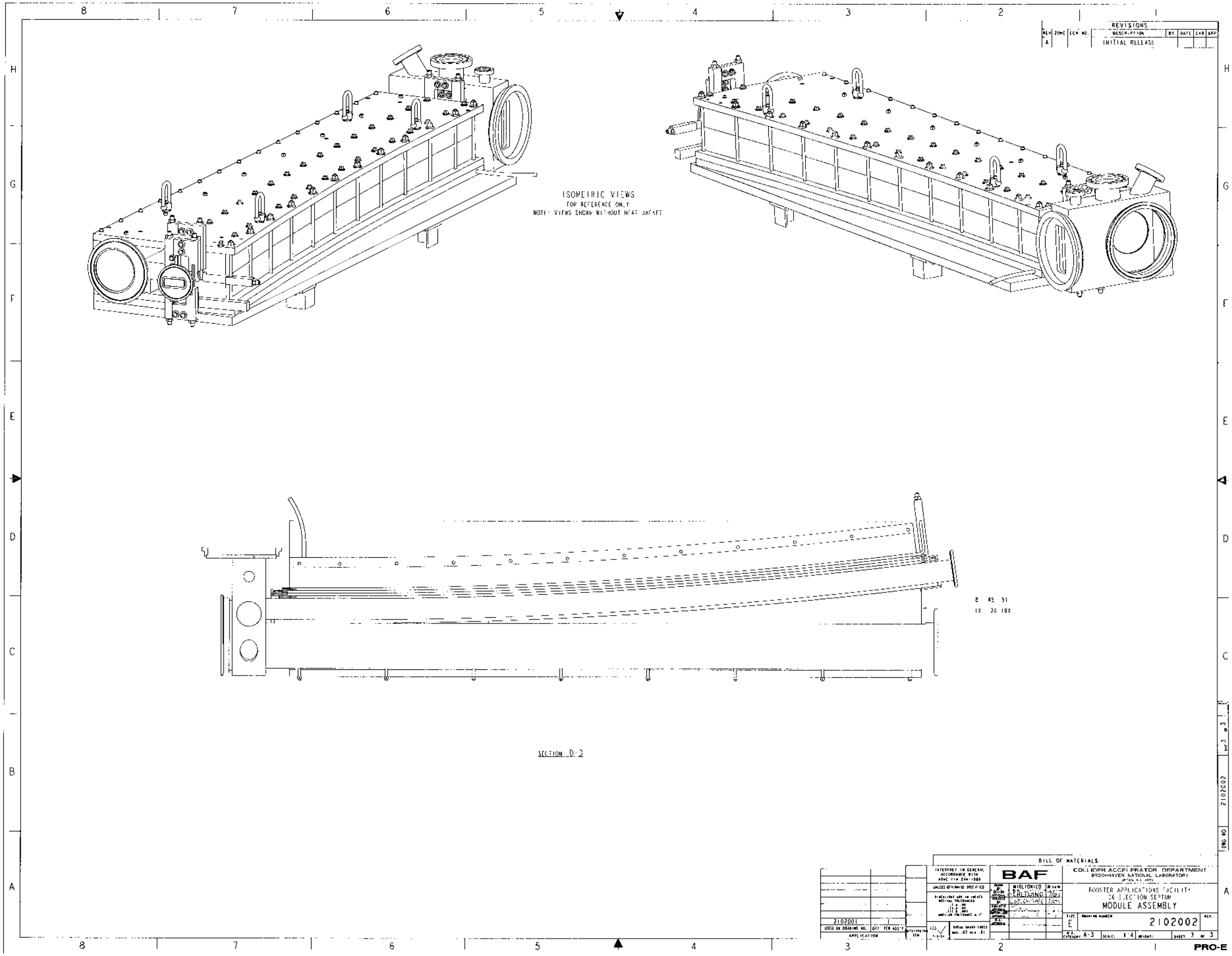
The D6 Septum Start of slides

Where is the D6 Septum Located?
What is its function ?

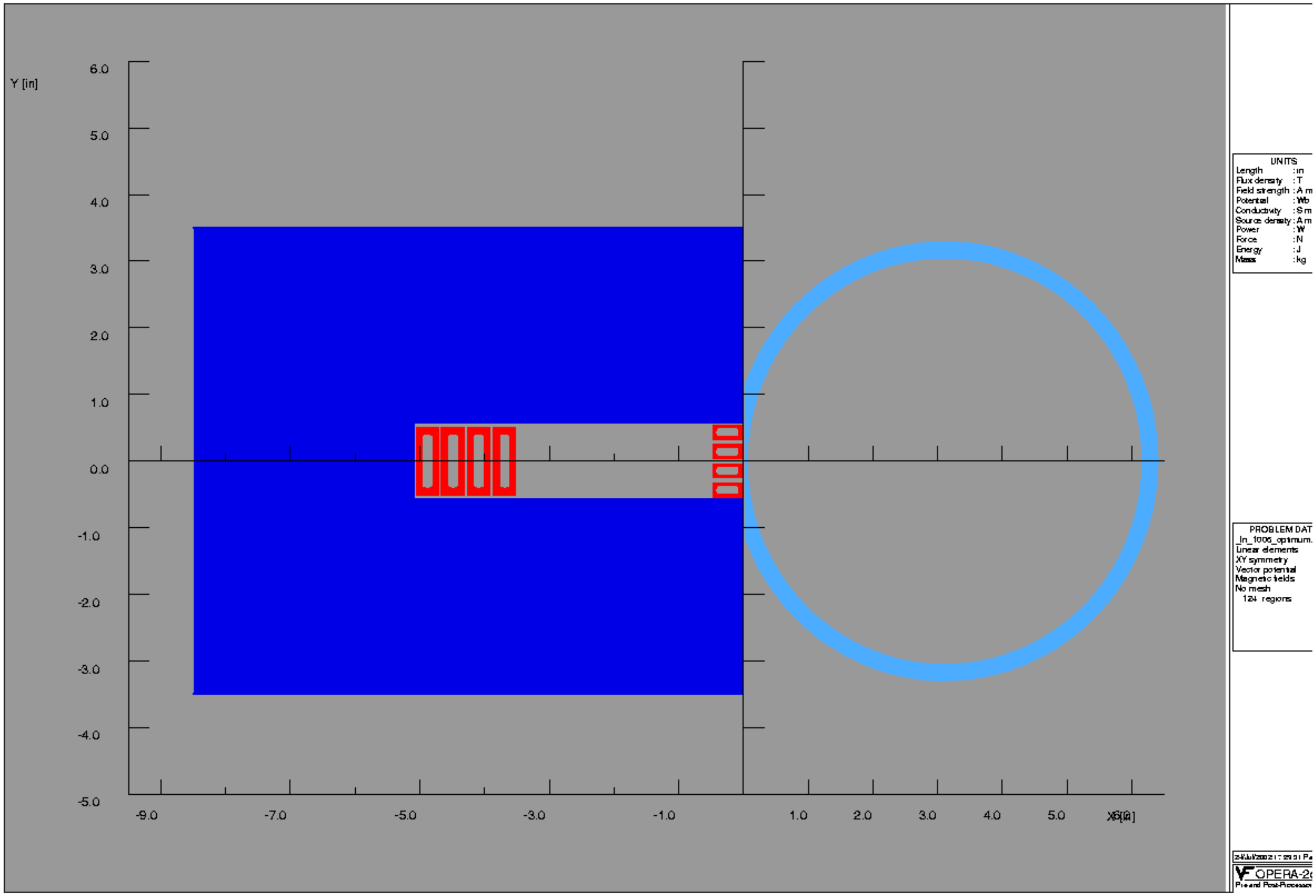
Isometric view of D6 Septum magnet



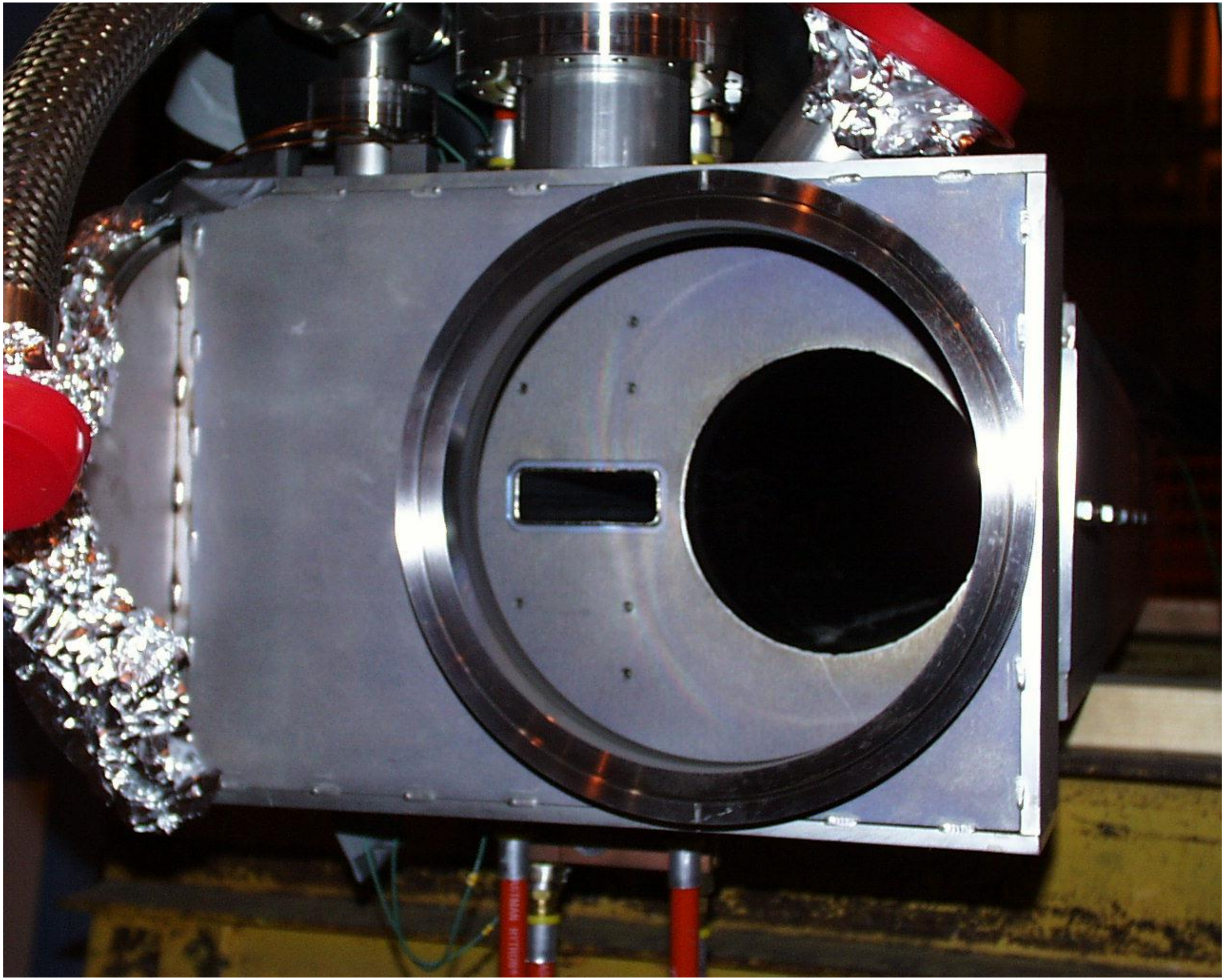
Isometric Views of D6-Septum Magnet



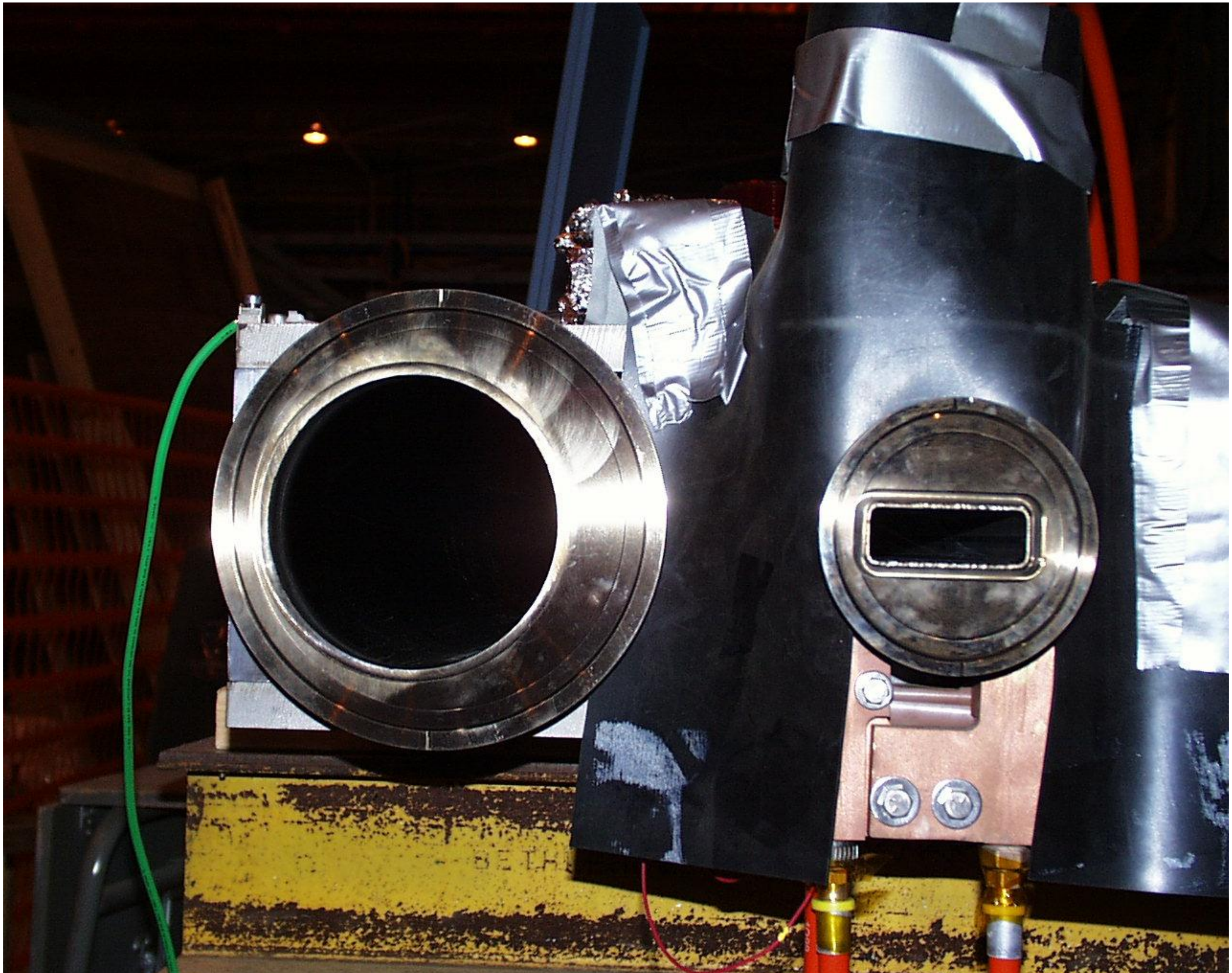
Cross section of the Septum Magnet at Beam Entrance (looking downstream). 2D Modeling



Cross section of the Septum Magnet at Beam Entrance (looking downstream). Picture by Ed Hoey

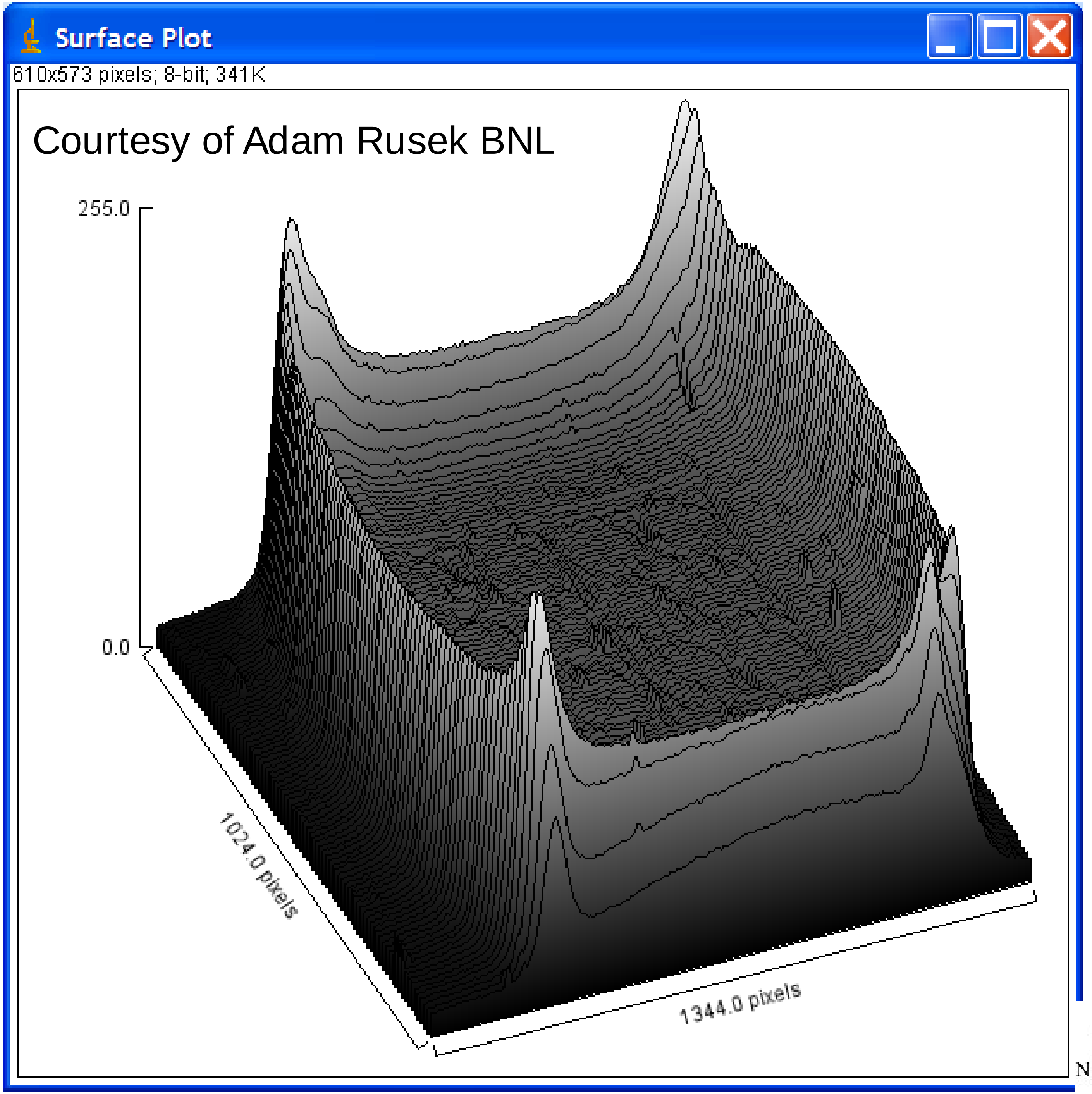


Cross section of the Septum Magnet at Beam Exit (looking upstream). Picture by Ed Hoey



The D6 Septum End of slides

Isometric 2D beam distribution at the NSRL Target



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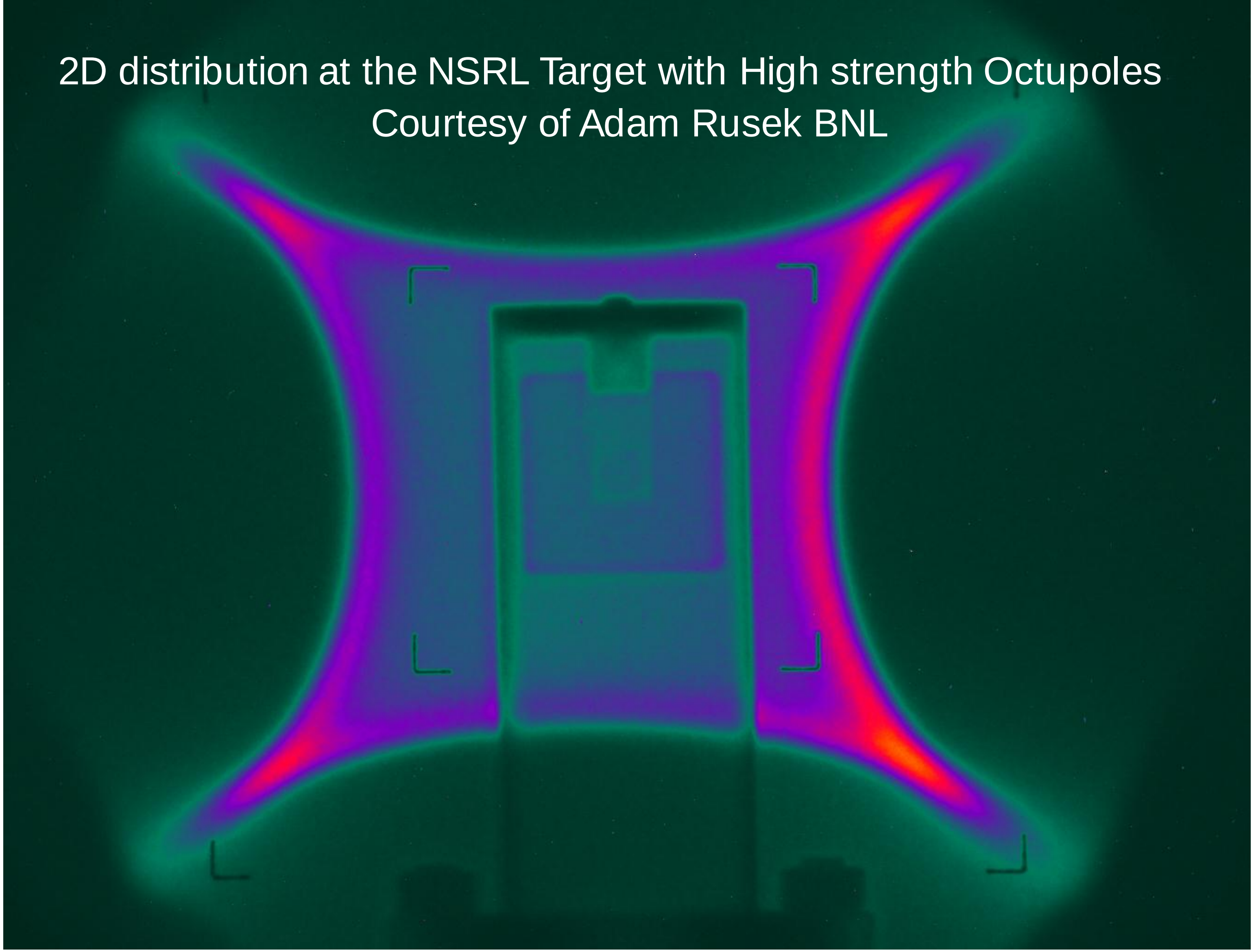
2D distribution at the NSRL Target with High strength Octupoles

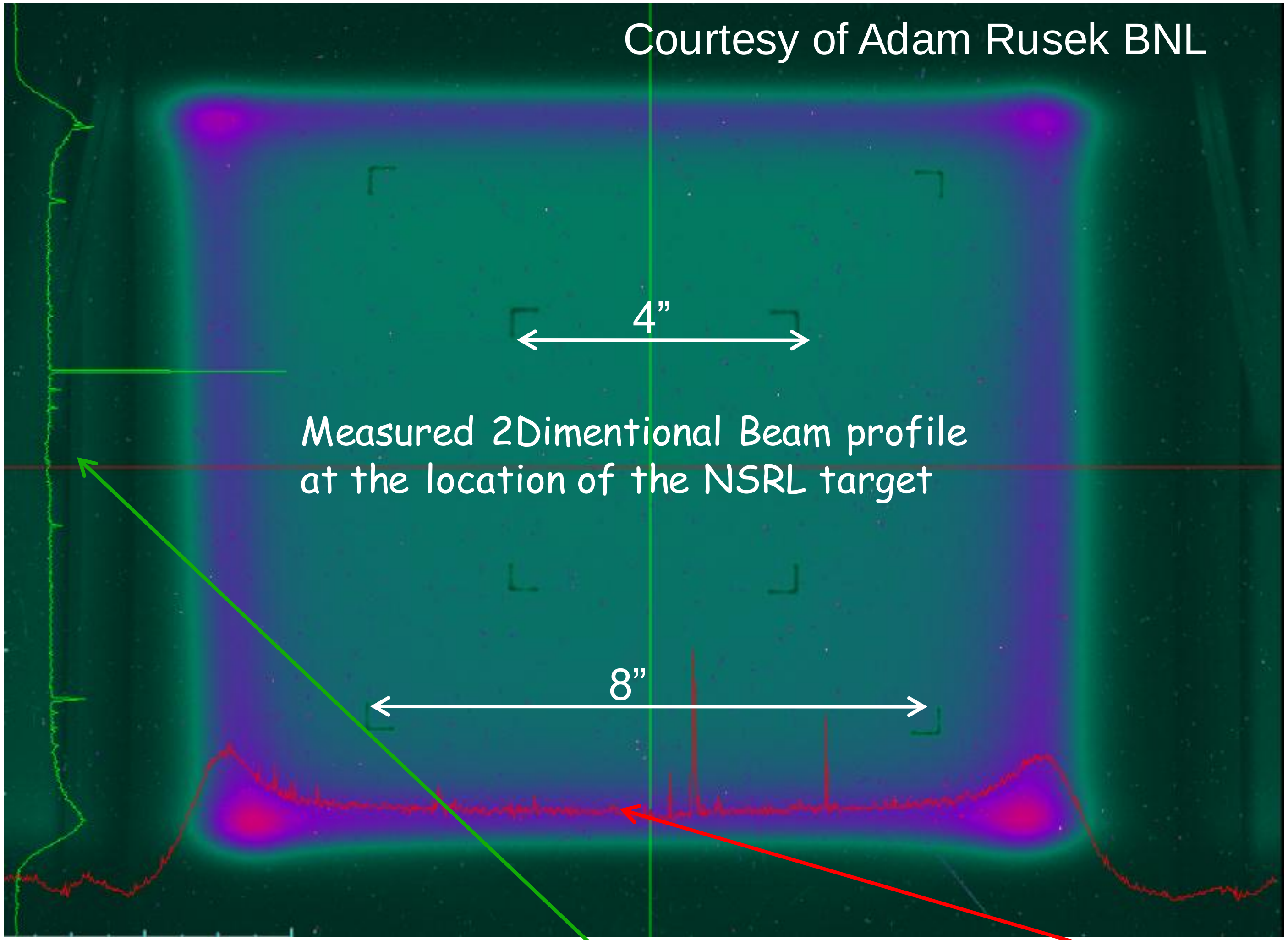
Courtesy of Adam Rusek BNL



$$B_y(r,\theta) = -\frac{B_0}{R^3} r^3 \cos(n\theta) = \frac{B_0}{R^3}(x^3 - 3xy^2)$$

$$B_x(r,\theta) = -\frac{B_0}{R^3} r^3 \sin(n\theta) = \frac{B_0}{R^3}(3x^2 y - y^3)$$

Experimental Results of Uniform beam Distributions at the target location of the NSRL facility at BNL



Measured Projected Vertical (Green line) and Horizontal (Red Line) Beam Profiles at the location of the NSRL target

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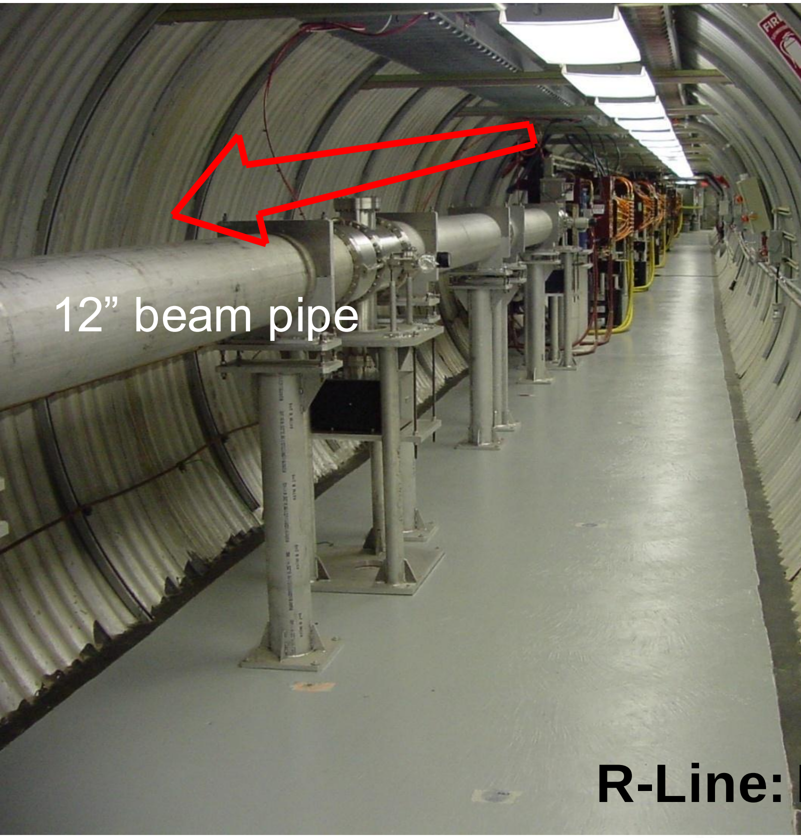
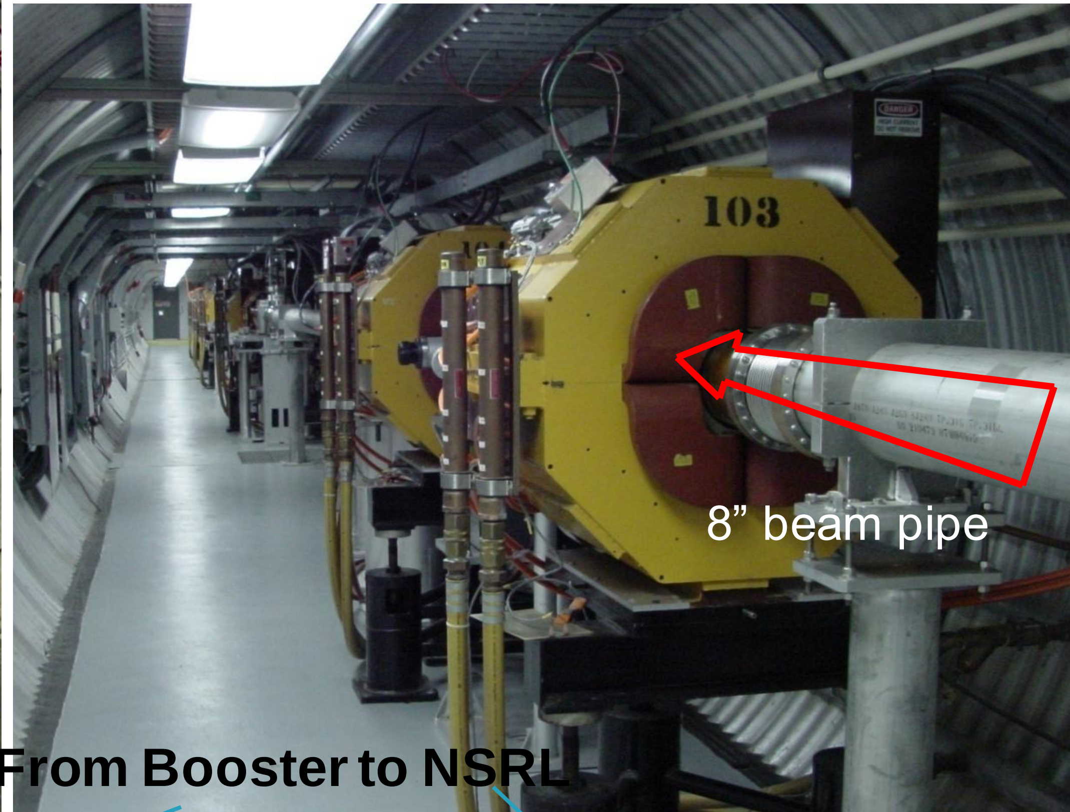
Do I need a computer code to generate a uniform distribution starting from a beam with normal distribution?

Is there a solution in closed form? even approximate?





R-Line: From Booster to NSRL